Orthonormal bases, orthogonal complements, and orthogonal direct sums

A sequence of vectors \( e_1, \ldots, e_n \) of an \( n \)-dimensional Euclidean space \( V \) is called an orthogonal basis, if it consists of nonzero vectors, which are pairwise orthogonal: \((e_i, e_j) = 0\) for \( i \neq j \). An orthogonal basis is called orthonormal, if all its vectors are of length 1.

**Lemma 1.** An orthogonal basis is a basis.

Indeed, assuming \( c_1e_1 + \ldots + c_ne_n = 0 \), we have

\[ 0 = (0, e_k) = (c_1e_1 + \ldots + c_ne_n, e_k) = c_1(e_1, e_k) + \ldots + c_n(e_n, e_k) = c_k(e_k, e_k), \]

which implies \( c_k = 0 \), since \( e_k \neq 0 \). (For any vector \( v \) we have \((0, v) = 0\) since \((0, v) = (2 \cdot 0, v) = 2(0, v)\).) Thus our system is linearly independent, and contains \( \dim V \) vectors, so is a basis.

**Lemma 2.** Any \( n \)-dimensional Euclidean space contains orthogonal bases.

We shall start from any basis \( f_1, \ldots, f_n \), and transform it into an orthogonal basis. Namely, we shall prove by induction that there exists a basis \( e_1, \ldots, e_k, f_{k+1}, \ldots, f_n \), where the first \( k \) vectors are pairwise orthogonal. Induction base is trivial, as for \( k = 1 \) there are no pairwise distinct vectors to be orthogonal, and we can put \( e_1 = f_1 \). Assume that our statement is proved for some \( k \), and let us show how to deduce it for \( k + 1 \). Let us search for \( e_{k+1} \) of the form \( f_{k+1} - a_1e_1 - \ldots - a_ke_k \). Conditions \((e_{k+1}, e_j) = 0\) for \( j = 1, \ldots, k \) mean that

\[ 0 = (f_{k+1} - a_1e_1 - \ldots - a_ke_k, e_j) = (f_{k+1}, e_j) - a_1(e_1, e_j) - \ldots - a_k(e_k, e_j), \]

and the induction hypothesis guarantees that the latter is equal to

\[ (f_{k+1}, e_j) - a_j(e_j, e_j), \]

so we can put \( a_j = \frac{(f_{k+1}, e_j)}{(e_j, e_j)} \). Let us show that the vector thus obtained is nonzero. From the very nature of our procedure, \( e_2 \) is a linear combination of \( f_1 \) and \( f_2, \ldots, e_k \) is a linear combination of \( f_1, \ldots, f_k \), so \( a_1e_1 + \ldots + a_ke_k \) is a linear combination of \( f_1, \ldots, f_k \), and

\[ f_{k+1} - a_1e_1 - \ldots - a_ke_k \neq 0 \]

since \( f_1, \ldots, f_n \) form a basis. This completes the proof of the induction step.

The procedure described above is called *Gram-Schmidt orthogonalisation procedure*. If after orthogonalisation we divide all vectors by their lengths, we obtain an orthonormal basis.
Lemma 3. For any inner product and any basis $e_1, \ldots, e_n$ of $V$, we have

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = \sum_{i,j=1}^{n} a_{ij}x_iy_j,$$

where $a_{ij} = (e_i, e_j)$.

This follows immediately from linearity property of inner products.

Corollary. A basis $e_1, \ldots, e_n$ is orthonormal if and only if

$$(x_1e_1 + \ldots + x_ne_n, y_1e_1 + \ldots + y_ne_n) = x_1y_1 + \ldots + x ny_n.$$
Due to orthonormality of our basis and the definition of the orthogonal complement, the left hand side of this equation is $c_j$. On the other hand, it is easy to see that for any $v$, the vector

$$v - (v, e_1)e_1 - \ldots - (v, e_k)e_k$$

is orthogonal to all $e_j$, and so to all vectors from $U$, and so belongs to $U^\perp$. The lemma is proved.

**Definition 2.** In the notation of the previous proof, $u$ is called the projection of $v$ onto $U$ and $u^\perp$ is called the perpendicular dropped from $v$ on $U$.

**Lemma 8.** $|u^\perp|$ is the shortest distance from the endpoint of $v$ to points of $U$:

$$|u^\perp| \geq |v - u_1|$$

for any $u_1 \in U$.

Indeed, $|v - u_1|^2 = |v - u + u - u_1|^2 = |v - u|^2 + |u - u_1|^2$ due to the Pythagoras theorem, so $|v - u_1|^2 \geq |v - u|^2$.

**Corollary (Bessel’s inequality).** For any vector $v \in V$ and any orthonormal system $e_1, \ldots, e_k$ (not necessarily a basis) we have

$$(v, v) \geq (v, e_1)^2 + \ldots + (v, e_k)^2.$$ 

Indeed, we can take $U = \text{span}(e_1, \ldots, e_k)$ and represent $v = u + u^\perp$.

Then $|v|^2 = |u|^2 + |u^\perp|^2 \geq |u|^2 = (u, e_1)^2 + \ldots + (u, e_k)^2 = (v, e_1)^2 + \ldots + (v, e_k)^2$.

**Example 1.** Consider the Euclidean space of all continuous functions on $[-\pi, \pi]$ with an inner product

$$(f(t), g(t)) = \int_{-\pi}^{\pi} f(t)g(t) \, dt.$$ 

It is easy to see that the functions

$$e_0 = \frac{1}{\sqrt{2\pi}}, e_1 = \frac{\cos t}{\sqrt{\pi}}, f_1 = \frac{\sin t}{\sqrt{\pi}}, \ldots, e_n = \frac{\cos nt}{\sqrt{\pi}}, f_n = \frac{\sin nt}{\sqrt{\pi}}$$

form an orthonormal system there. Consider the function $h(t) = t$. We have

$$(h(t), h(t)) = \frac{2\pi^3}{3},$$

$$(h(t), e_0) = 0,$$

$$(h(t), e_k) = 0,$$

$$(h(t), f_k) = \frac{2(-1)^{k+1}\sqrt{\pi}}{k},$$
(the latter integral requires integration by parts to compute it), so Bessel’s inequality implies that

\[ \frac{2\pi^3}{3} \geq 4\pi + \frac{4\pi}{4} + \frac{4\pi}{9} + \ldots + \frac{4\pi}{n^2}, \]

which can be rewritten as

\[ \frac{\pi^2}{6} \geq 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}. \]

Actually \( \sum_k \frac{1}{k^2} = \frac{\pi^2}{6} \), which was first proved by Euler. We are not able to establish it here, but it is worth mentioning that Bessel’s inequality gives a sharp bound for this sum.