

1S11: CALCULUS FOR STUDENTS IN SCIENCE

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TCD

Lecture 7

INVERSE FUNCTIONS: REMINDER

Definition. Suppose that for a function f there exists a function g such that

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g,$$

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

Then g is said to be the inverse of f , f is said to be the inverse of g , and it is always said that f and g are inverse functions.

Using the notation f^{-1} for the inverse, the definition can be rewritten as

$$f(f^{-1}(x)) = x \text{ for all } x \text{ in the domain of } f^{-1},$$

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

INVERSE FUNCTIONS

Suppose that the equation $y = f(x)$ can be solved for x as a function of y , say $x = g(y)$.

Then we can substitute $x = g(y)$ into $f(x) = y$, and obtain $f(g(y)) = y$.

Also we can substitute $y = f(x)$ into $x = g(y)$, and obtain $g(f(x)) = x$.

Therefore g is the inverse function of f .

INVERSE FUNCTIONS

This leads to the following **procedure for computing inverse functions**:

- Form the equation $y = f(x)$.
- If possible, solve that equation for x as a function of y , $x = g(y)$.
- If x is required as the name of the independent variable, interchange x and y : $y = g(x)$. This gives the inverse function, when it exists.

INVERSE FUNCTIONS

Example. Let $f(x) = \sqrt{3x - 2}$. We form the equation $y = \sqrt{3x - 2}$. Solving this equation for x goes as follows:

$$y = \sqrt{3x - 2},$$

$$y^2 = 3x - 2,$$

$$x = \frac{y^2 + 2}{3}.$$

Interchanging names of variables, we obtain the formula $g(x) = \frac{x^2 + 2}{3}$. Note that we proved already that the domain of f^{-1} is equal to the range of f . The range of $\sqrt{3x - 2}$ is manifestly $[0, +\infty)$, so we get the inverse

$$f^{-1}(x) = \frac{x^2 + 2}{3}, \quad x \geq 0.$$

INVERSE FUNCTIONS

The procedure for computing inverses that we described only works when the equation $y = f(x)$ has one solution for each value of y , otherwise, it is not possible to assign a well defined value to x . Indeed, if $f(a) = y = f(b)$, then

$$a = f^{-1}(f(a)) = f^{-1}(y) = f^{-1}(f(b)) = b,$$

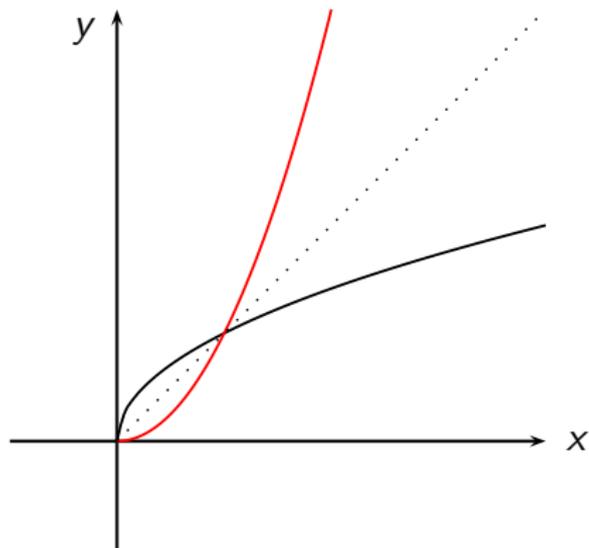
a contradiction.

This leads to the *horizontal line test* for invertibility: a function is invertible if and only if each horizontal line meets its graph at most once.

Algebraically, one can say that f is invertible if and only if it is “one-to-one”, that is it assigns different values of y to different values of x .

INVERSE FUNCTIONS

Also, since f^{-1} is obtained from f by switching roles of x and y , the graph of f^{-1} is obtained from the graph of f by switching the x -axis and the y -axis, that is the symmetry about the line $y = x$:



In the view of this, the *horizontal line test* is merely the *vertical line test* after swapping x and y .

INVERSE FUNCTIONS

A function f is said to be (strictly) increasing if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$. Similarly, a function f is said to be (strictly) decreasing if $f(x_1) < f(x_2)$ whenever $x_1 > x_2$.

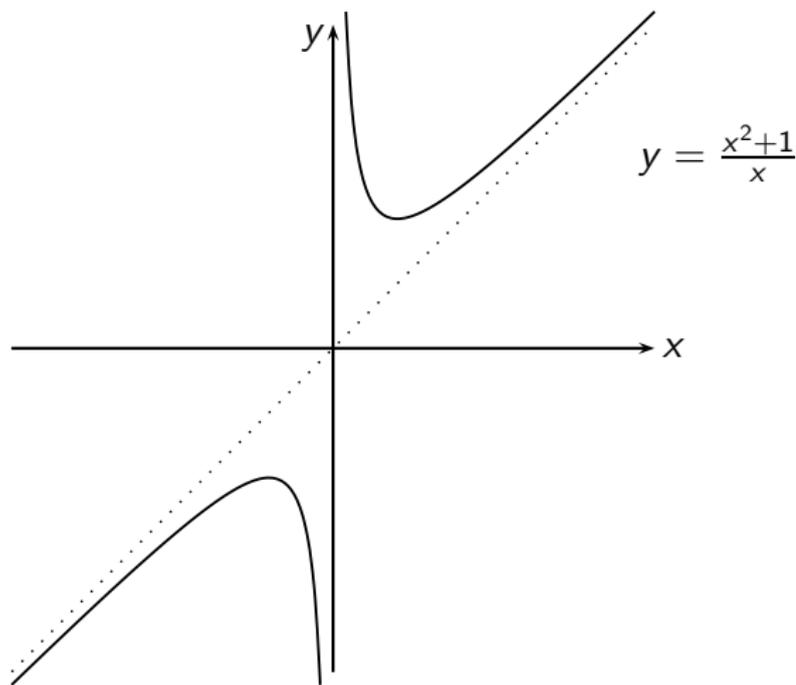
Manifestly, a function which is strictly increasing or strictly decreasing is one-to-one, so it is invertible.

SUMMARY

This is what we discussed while dealing with functions:

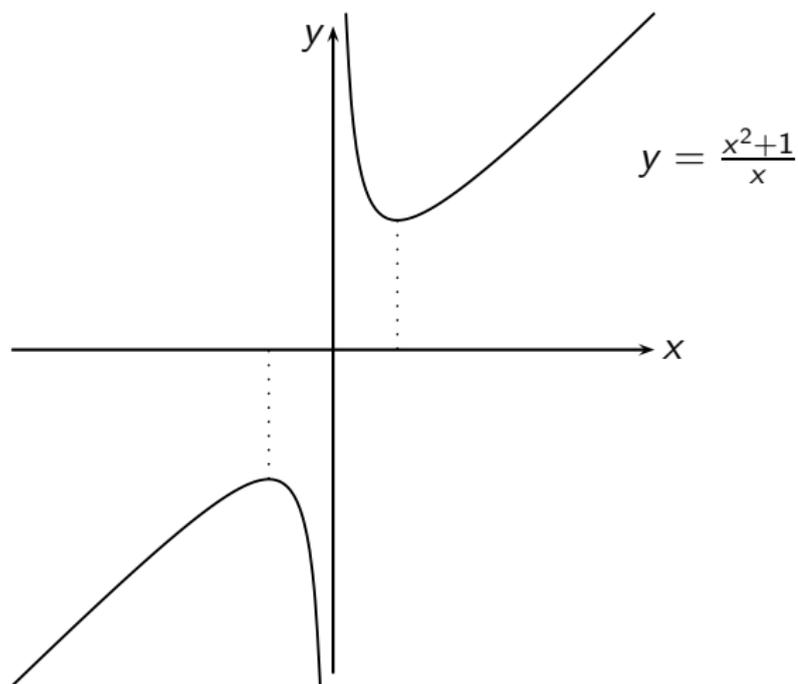
- The notions of *domain* and *range* of a function.
- Graph of a function, and the vertical line test.
- Arithmetic operations and composition.
- Elementary changes of graphs: translation, reflection, scaling.
- Symmetries of graphs and of plane curves.
- Classical functions and families of functions.
- Graphs of various classical functions. Asymptotes.
- Inverse functions, their domains and ranges. The horizontal line test.
Graphs of inverse functions.

WHAT NEXT?



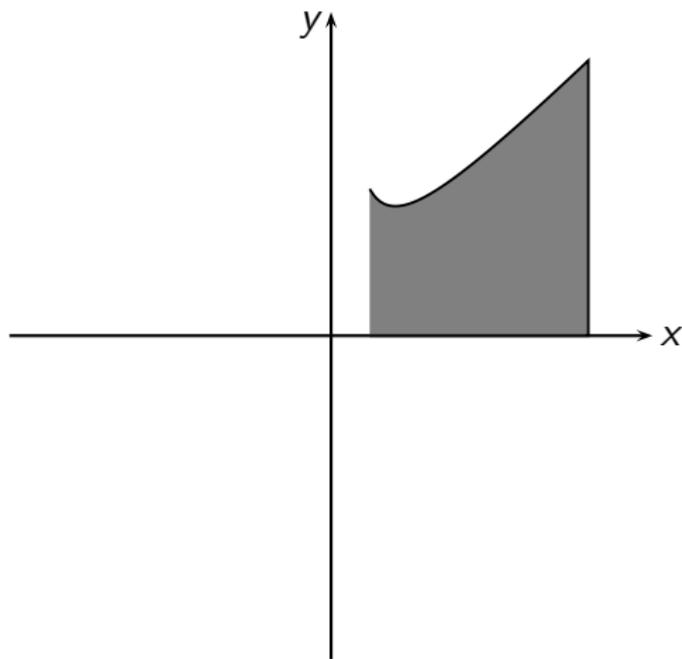
As we discussed in Lecture 5, the graph of $\frac{x^2+1}{x}$ has a vertical asymptote $x = 0$, and no horizontal asymptotes. It however has another straight line asymptote $y = x$, because $\frac{x^2+1}{x} = x + \frac{1}{x}$.

WHAT NEXT?



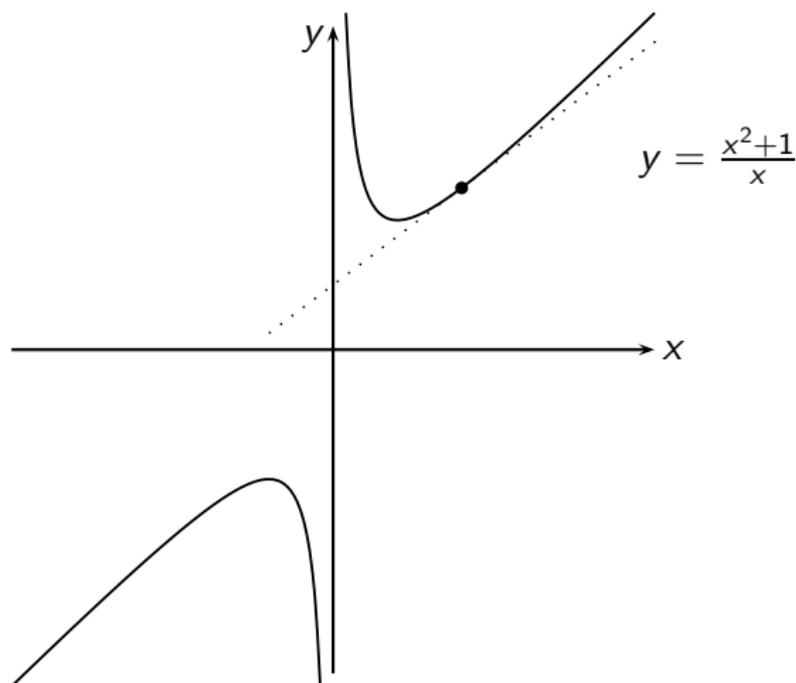
What about the points where values are (“locally”) maximal or minimal?

WHAT NEXT?



What about the area under a graph?

WHAT NEXT?



What about the tangent lines to a graph? How to write their equations?

WHAT NEXT?

- Things related to computing tangent lines are dealt with by “differential calculus”. (Relates to the notion of *derivative*.)
- Things related to computing areas are dealt with by “integral calculus”. (Relates to the notion of *antiderivative*.)
- In fact, these two problems are strongly related, and one can be reduced to another (for appropriate graphs). (Relates to “Newton–Leibniz formula”, sometimes called “the fundamental theorem of calculus”.)

Our goal for the rest of this semester is to elaborate on those three points, and learn to use the respective methods.

EXAMPLE FROM DIFFERENTIAL CALCULUS

The tangent line to $y = x^2$ at the point $(1, 1)$ can be obtained as the limiting positions of the secant lines passing through the points $(1, 1)$ and $(1 + h, (1 + h)^2)$ as h gets smaller and smaller. Lines passing through $(1, 1)$ have the equations

$$y - 1 = m(x - 1),$$

where m is the slope. If $(1 + h, (1 + h)^2)$ is on this line, we get

$$(1 + h)^2 - 1 = m(1 + h - 1),$$

or $2h + h^2 = mh$, which for $h \neq 0$ simplifies to $m = 2 + h$. As h becomes smaller and smaller, this value of m gets closer and closer to 2, so secant lines get closer and closer to the line $y - 1 = 2(x - 1)$, or $y = 2x - 1$.