1S11: Calculus for students in Science

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TCD

Lecture 32

Work and integration

In the previous classes, we already discussed in passing some instances when work is performed by a force on some object. Now we shall discuss that in more detail.

Basic principle. If a constant force of magnitude F is applied in the direction of the motion of an object, and the object moves a distance d under the action of that force, then the work performed by that force on the object is defined to be

$$W = F \cdot d.$$

Using that principle, one can also define and compute the work when the force is changing depending on the position of an object. Of course, the method in this case is our usual method: divide the path of the object into many very small parts, assume the force to be constant on each part, and add the results together.

Work and integration

As a consequence, the work will be equal to the appropriate Riemann sum

$$W = \sum_{i=1}^{N} F(x_i^*)\Delta x_i.$$

As the mesh size of the partition tends to zero, this quantity has the limit

$$\int_a^b F(x)\,dx,$$

where a and b are the initial position and the final position of the object, respectively.

Example (Hooke's Law)

Hooke's Law states that a spring stretched x units beyond its natural length pulls back with a force

$$F(x) = kx$$

where k is a constant called the *stiffness* of the spring; it depends on the material as well as the thickness of the spring.

Example. Suppose that a spring exerts a force of 5 N when stretched one metre beyond its natural length. Find the work required to stretch the spring 1.8 metres beyond its natural length.

Solution. Let us first compute the stiffness of this spring. Applying Hooke's Law with $F = 5$, $x = 1$, we get $k = 5$ (N/m). Now, the work required is

$$W = \int_a^b F(x)\,dx = \int_0^{1.8} 5x\,dx = \left.\frac{5x^2}{2}\right]_0^{1.8} = 8.1\,(\mathrm{N}\cdot\mathrm{m}).$$

Relationship between work and energy

Let us assume that an an object of mass m moves along the x axis as a result of the force $F(x)$ that is applied in the direction of motion. As time passes, the acceleration, as a function of time, is the rate of change of the instantaneous velocity of an object. Suppose that the object is at the position $x(t)$ at the time t, so that $x'(t) = v(t)$ is the instantaneous velocity, and $v'(t) = a(t)$ is the instantaneous acceleration.

Newton's Second Law of Motion. If an object of mass m is moving as a result of a force F applied to it, then that object undergoes an acceleration a that satisfies the equation

$$F = ma.$$

Relationship between work and energy

Suppose that at the initial moment t_0 the object is at the position $x(t_0) = a$ moving with the initial velocity v_i, and at the final moment t_1 the object is at the position $x(t_1) = b$ moving with the final velocity v_f.

The work of the force moving the object is $\int\limits_a^b F(x)\,dx$, which we can rewrite as

$$\int_{x(t_0)}^{x(t_1)} F(x)\,dx \overset{x=x(t),\,dx=x'(t)\,dt}{=} \int_{t_0}^{t_1} F(x(t))x'(t)\,dt =$$

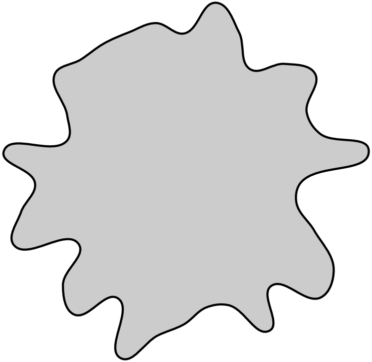
$$= \int_{t_0}^{t_1} ma(t)v(t)\,dt = \int_{t_0}^{t_1} mv'(t)v(t)\,dt \overset{v=v(t),\,dv=v'(t)\,dt}{=}$$

$$= \int_{v(t_0)}^{v(t_1)} mv\,dv = \left.\frac{mv^2}{2}\right]_{v_i}^{v_f} = \frac{mv_f^2}{2} - \frac{mv_i^2}{2}.$$

The quantity $\frac{mv^2}{2}$ is usually referred to as the *kinetic energy* of an object. We just established that the work performed by the force on the object is equal to the change in the kinetic energy of the object.

Centre of gravity of a lamina

By a lamina, I shall mean a flat object thin enough to be viewed as a 2d plane region.



Centre of gravity of a lamina

A lamina will be assumed *homogeneous*, that is being composed uniformly throughout. The *density* δ of a lamina is its mass per unit area.

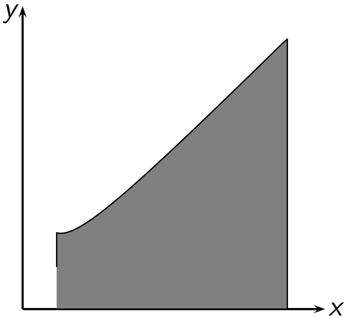
It can be shown that for each lamina, it is possible to find a point $(\bar{x}, \bar{y})$ such that the effect of gravity on the lamina is equivalent to that of a single force activng at the point $(\bar{x}, \bar{y})$. This point is called the *centre of gravity* of the lamina. For a symmetric lamina, like a circle, or a square, the centre of gravity coincides with the symmetry centre, but for a more complex shape it is not as obvious.

From basic mechanics, one can demonstrate that for a lamina whose mass is localised at finitely many points $A_1, \ldots, A_n$ (with masses $m_1, \ldots, m_n$ respectively), its centre of gravity M can be determined from the so called *equilibrium conditions*

$$m_1(\overrightarrow{OA_1} - \overrightarrow{OM}) + m_2(\overrightarrow{OA_2} - \overrightarrow{OM}) + \cdots + m_n(\overrightarrow{OA_n} - \overrightarrow{OM}) = 0.$$

Centre of gravity of a lamina

We shall compute the centre of gravity of a lamina occupying the region bounded by a graph $y = f(x)$ and the x-axis (on a finite interval $[a, b]$).



Centre of gravity of a lamina

Let us divide the interval $[a, b]$ in many small parts, approximating the lamina by a union of rectangles. The centre of gravity of each individual rectangle is at the point $(x_k^*, \frac{1}{2}f(x_k^*))$, where x_k^* is the midpoint of its base. If we denote by Δm_k the mass of the k-th rectangle, the gravity centre equilibrium conditions are

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\Delta m_k = 0,$$

$$\sum_{k=1}^{n}(y_k^* - \bar{y})\Delta m_k = 0,$$

where $\Delta m_k = \delta f(x_k^*)\Delta x_k$ and $y_k^* = \frac{1}{2}f(x_k^*)$, so

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k = 0,$$

$$\sum_{k=1}^{n}\left(\frac{1}{2}f(x_k^*) - \bar{y}\right)\delta f(x_k^*)\Delta x_k = 0.$$

Centre of gravity of a lamina

As the mesh size of the partition of $[a, b]$ gets smaller, the equations

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\delta f(x_k^*)\Delta x_k = 0,$$

$$\sum_{k=1}^{n}\left(\frac{1}{2}f(x_k^*) - \bar{y}\right)\delta f(x_k^*)\Delta x_k = 0$$

become

$$\int_a^b (x-\bar{x})\delta f(x)\,dx = 0, \qquad \int_a^b \left(\frac{1}{2}f(x) - \bar{y}\right)\delta f(x)\,dx = 0.$$

Recalling that $\bar{x}$ and $\bar{y}$ are constants, these can be written as

$$\int_a^b \delta x f(x)\,dx = \bar{x}\int_a^b \delta f(x)\,dx, \qquad \int_a^b \frac{1}{2}\delta(f(x))^2\,dx = \bar{y}\int_a^b \delta f(x)\,dx.$$

Centre of gravity of a lamina

Examining the formulas

$$\int_a^b \delta x f(x)\,dx = \bar{x}\int_a^b \delta f(x)\,dx, \qquad \int_a^b \frac{1}{2}\delta (f(x))^2\,dx = \bar{y}\int_a^b \delta f(x)\,dx$$

more carefully, we note that the factor δ can be dropped since the lamina is homogeneous, and it is just a constant, and that $\int\limits_a^b f(x)\,dx$ is the area of the lamina, so we get the formulas

$$\bar{x} = \frac{\int_a^b x f(x)\,dx}{\text{area of the lamina}},$$
$$\bar{y} = \frac{\int_a^b \frac{1}{2}(f(x))^2\,dx}{\text{area of the lamina}}.$$

Centre of gravity of a lamina

Example. Assume that the lamina is a half-circle $0 \le y \le \sqrt{1-x^2}$. We have

$$\bar{x} = \frac{\int_{-1}^{1} x\sqrt{1-x^2}\,dx}{\text{area of the lamina}},$$
$$\bar{y} = \frac{\int_{-1}^{1} \frac{1}{2}(1-x^2)\,dx}{\text{area of the lamina}}.$$

Clearly, $\bar{x}$ is proportional to the integral of an odd function, and is therefore equal to zero. As for $\bar{y}$, we have

$$\bar{y} = \frac{\int_{-1}^{1} \frac{1}{2}(1-x^2)\,dx}{\frac{1}{2}\pi} = \frac{x - \frac{x^3}{3}\Big]_{-1}^{1}}{\pi} = \frac{1-\frac{1}{3}+1-\frac{1}{3}}{\pi} = \frac{4}{3\pi}.$$

That's it with calculus for this semester. Thank you for your attention, and do not forget to check the module webpage

`http://www.maths.tcd.ie/~vdots/teaching/1S11-1314.html`

for sample exam problems to practise!

Merry Christmas!