1S11: Calculus for students in Science

Dr. Vladimir Dotsenko

TCD

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Substitutions in definite integrals

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Recall the u-substitution method for computing antiderivatives: given an integral of the form

$$\int f(g(x))g'(x)\,dx,$$

we denote $u = g(x)$ so that $du = g'(x)\,dx$, so that the integral becomes

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$$\int_a^b f(g(x))g'(x)\,dx,$$

we need to take appropriate care of the effect of that on the limits of integration.

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we need to take appropriate care of the effect of that on the limits of integration. There are two ways to deal with it, which we shall now outline.

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Method 1. Use u-substitutions only on the level of indefinite integrals: evaluate

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and then use the formula

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Method 2. Use the relationship $u = g(x)$ to modify the limits:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

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$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C.$$

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Therefore,

$$\int_0^2 x(x^2+1)^3\,dx = \left[\int x(x^2+1)^3\,dx\right]_0^2 = \left[\frac{(x^2+1)^4}{8}\right]_0^2 = \frac{625}{8} - \frac{1}{8} = 78.$$

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Therefore,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\left(\frac{5^4}{4} - \frac{1^4}{4}\right) = 78.$$

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Therefore,

$$\int_1^3 \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int_\pi^{\pi/3} \cos u\,du =$$
$$= -\frac{1}{\pi}\sin u\Big]_\pi^{\pi/3} = -\frac{1}{\pi}(\sin(\pi/3) - \sin\pi) = -\frac{\sqrt{3}}{2\pi} \approx -0.276.$$

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Therefore,

$$\int_0^{\pi/4} \sqrt{\tan x}\frac{1}{\cos^2 x}\,dx = \int_0^1 \sqrt{u}\,du = \left.\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}.$$

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$$\int_0^{\pi/2} \cos^n x\, dx = -\int_{\pi/2}^{0} \sin^n u\, du = \int_0^{\pi/2} \sin^n x\, dx,$$

as required.

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$$\text{for } x = -1, u = -1, \quad \text{and for } x = 1, u = 1,$$

and that $\frac{1}{1+x^2} = \frac{1}{1+(1/u)^2} = \frac{u^2}{1+u^2}$.

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$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = -\int_{-1}^{1} \frac{u^2}{1+u^2}\frac{1}{u^2}\,du = -\int_{-1}^{1} \frac{1}{1+u^2}\,du,$$

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so the integral is equal to its negative and hence equal to zero. How is it possible? Of course, it happened because $u = g(x)$ was not defined on all the interval $[-1, 1]$, having a singularity at $x = 0$.

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Let us evaluate the integral

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and transform the second integral using the substitution $u = \frac{\pi}{2} - x$, so that $du = -dx$,

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$\cos(x) = \cos(\pi/2 - u) = \sin u$, leading to

$$\int_0^{\pi/2} \cos^2(\cos x)\,dx = -\int_{\pi/2}^{0} \cos^2(\sin u)\,du = \int_0^{\pi/2} \cos^2(\sin x)\,dx.$$

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Therefore,

$$\int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x))\,dx =$$

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$$= \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

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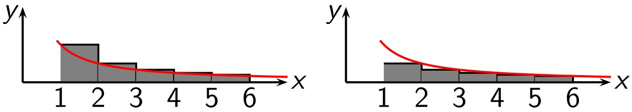

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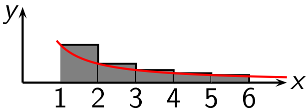

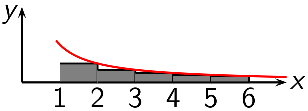

These two figures prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln(1) = \ln(n+1),$$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} \leq \int_1^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln(1) = \ln(n+1),$$

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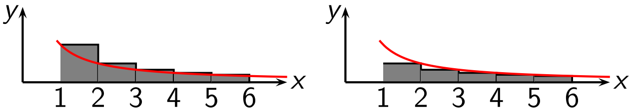

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so we have

$$\ln(n+1) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln(n+1) + 1 - \frac{1}{n+1}.$$

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Next time: applications of the definite integral in geometry, science, and engineering.