1S11: Calculus for students in Science

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TCD

Lecture 3

New functions from old ones: composition (reminder slide)

Definition. The *composition* of two functions f and g, denoted by $f \circ g$, is the function whose value at x is $f(g(x))$:

$$(f \circ g)(x) = f(g(x)).$$

Its domain is defined as the set of all x in the domain of g for which the value $g(x)$ is in the domain of f.

Example 1. Completing the square for solving quadratic equations:

$$x^2+px+q = x^2+2\frac{p}{2}x+q = x^2+2\frac{p}{2}x+\frac{p^2}{4}-\frac{p^2}{4}+q = \left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+q,$$

represents x^2+px+q as the composition $f(g(x))$, where $f(x) = x^2 - \left(\frac{p^2}{4} - q\right)$ and $g(x) = x + \frac{p}{2}$.

Example 2. Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Then

$$(f \circ g)(x) = (\sqrt{x})^2 + 3 = x + 3.$$

Note that the domain of f is $(-\infty, +\infty)$, and the domain of g is $[0, +\infty)$, so the only restriction we impose on x to get the domain of $f \circ g$ is that g is defined, and we conclude that $(f \circ g)(x) = x + 3, x \geq 0$.

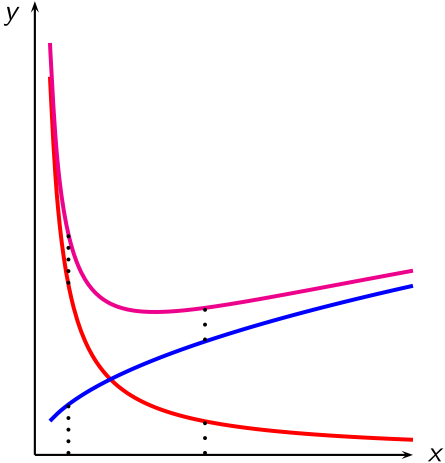
On the other hand,

$$(g \circ f)(x) = \sqrt{x^2 + 3}.$$

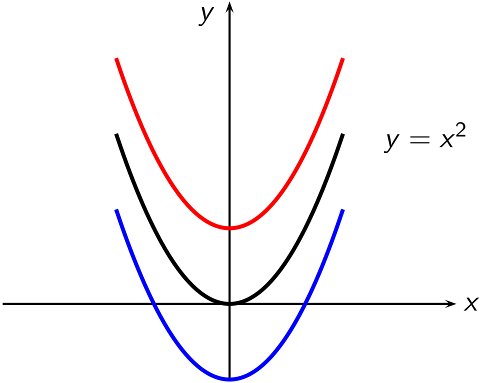
Since f is defined everywhere, the only restriction we impose on x to get the domain of $g \circ f$ is that $f(x)$ is in the domain of g, so since $x^2 + 3$ is positive for all x, we conclude that $(g \circ f)(x) = \sqrt{x^2 + 3}$. (With its natural domain $(-\infty, +\infty)$).

New functions from old ones: examples

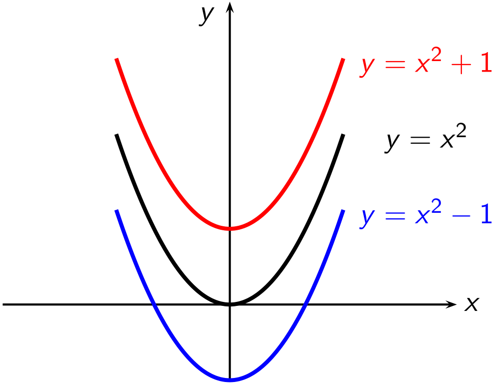
Let us plot some graphs to get a better feeling on how operations on functions work. To begin with, we obtain the graph of $f(x) = \sqrt{x} + \frac{1}{x}$ from the graphs of of $\sqrt{x}$ and $1/x$.



New functions from old ones: examples

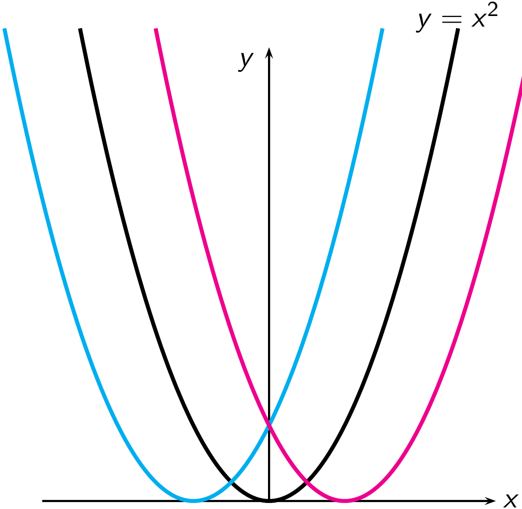


New functions from old ones: examples

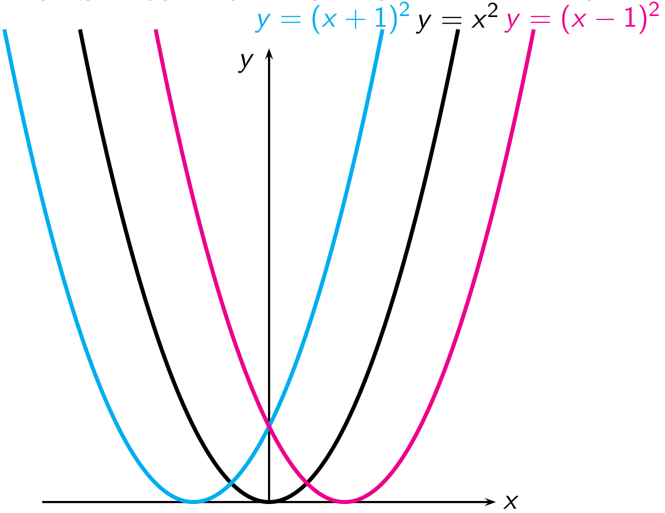


Let t denote the translation function, $t(x) = x + a$. Replacing $f(x)$ by $f(x) + a = (t \circ f)(x)$ shifts the graph vertically: up if $a > 0$, down if $a < 0$.

New functions from old ones: examples



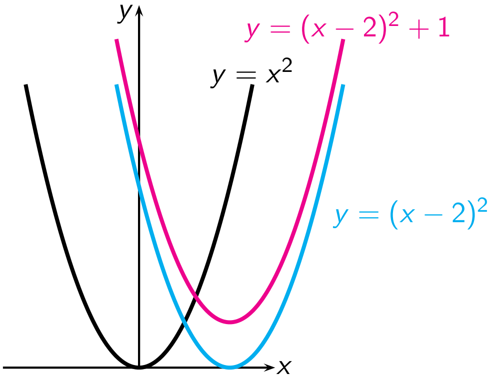
New functions from old ones: examples



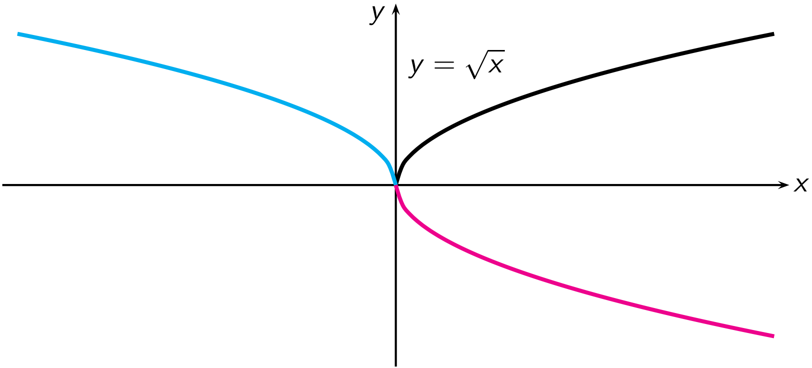
Let t denote the translation function, $t(x) = x + a$. Replacing $f(x)$ by $f(x+a) = (f \circ t)(x)$ shifts the graph horizontally: left if $a > 0$, right if $a < 0$.

New functions from old ones: examples

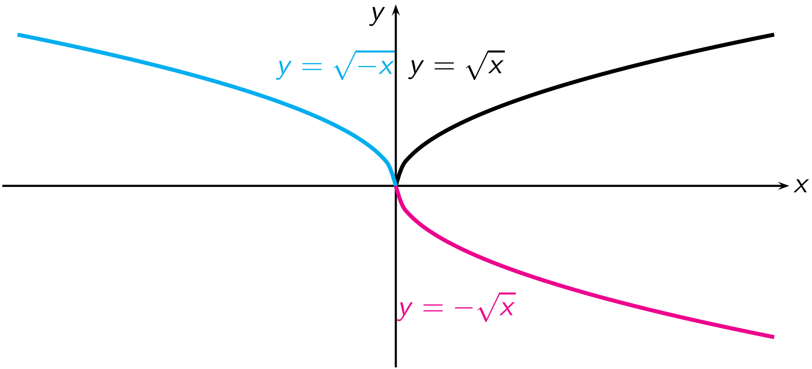
Let us plot the graph of the function $y = x^2 - 4x + 5$. By completing the square, we obtain $y = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$.



New functions from old ones: examples



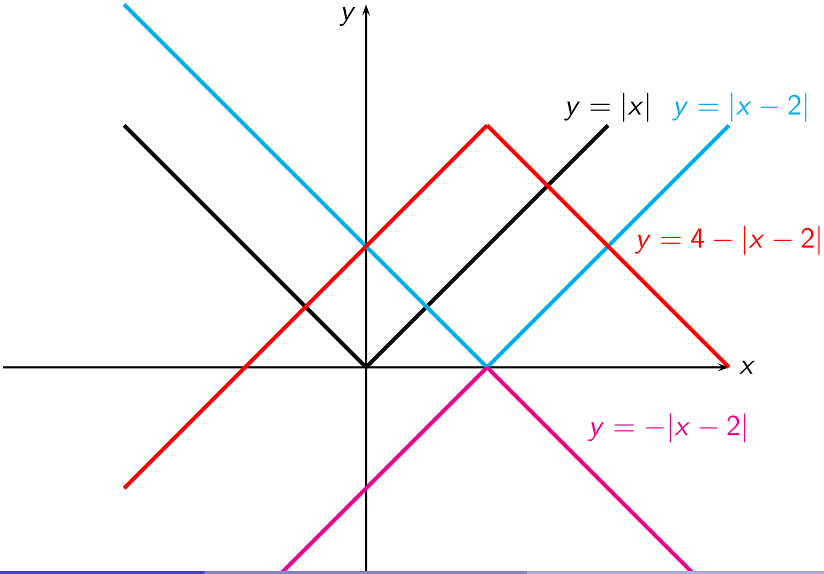
New functions from old ones: examples



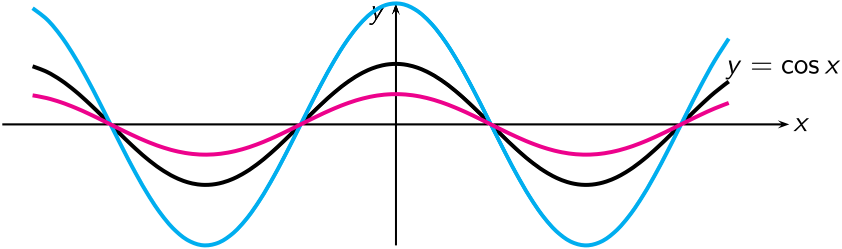
Let r denote the reflection function, $r(x)=-x$. Replacing a function $f(x)$ by $f(-x)=(f\circ r)(x)$ reflects the graph about the y-axis, and replacing $f(x)$ by $-f(x)=(r\circ f)(x)$ reflects the graph about the x-axis.

New functions from old ones: examples

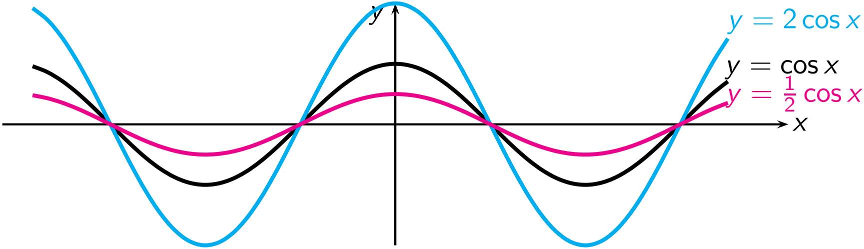
Let us plot the graph of the function $y = 4 - |x-2|$.



New functions from old ones: examples

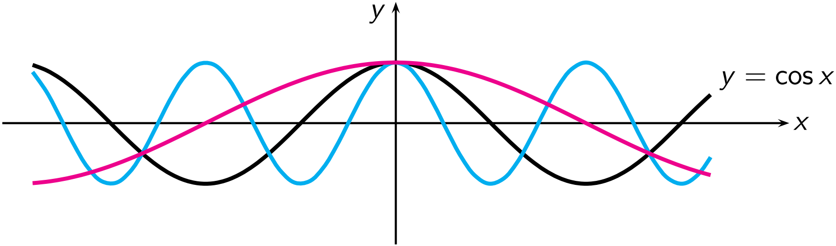


New functions from old ones: examples

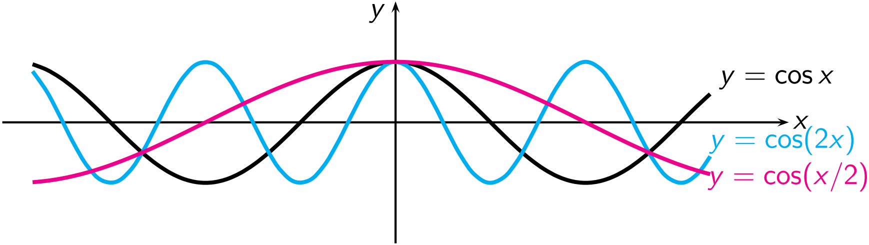


Let s denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $cf(x) = (s \circ f)(x)$ stretches the graph vertically if $c > 1$, and compresses the graph vertically if $0 < c < 1$.

New functions from old ones: examples



New functions from old ones: examples



Let s denote the scaling function, $s(x) = cx$. Replacing a function $f(x)$ by $f(cx) = (f \circ s)(x)$ compresses the graph horisontally if $c > 1$, and stretches the graph horisontally if $0 < c < 1$.