1S11: Calculus for students in Science

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TCD

Lecture 20

Derivatives and analysis of functions: reminder

The following facts will be useful for us. We shall use them without proof. The maximal generality in which we shall use these statements would be for a function f that is continuous on a closed interval $[a, b]$ and differentiable on the corresponding open interval (a, b).

- If f is a constant function on $[a, b]$, then $f'(x) = 0$ for all x in (a, b).
- If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.
- If f is increasing on $[a, b]$, then $f'(x) \geq 0$ for all x in (a, b).
- If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).
- If f is decreasing on $[a, b]$, then $f'(x) \leq 0$ for all x in (a, b).
- If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

Examples

Example 1. Let us consider the function $f(x) = x^2 - 6x + 5$. Its derivative $f'(x) = 2x - 6 = 2(x - 3)$, so $f'(x) < 0$ for $x < 3$, and $f'(x) > 0$ for $x > 3$. We conclude that f is decreasing on $(-\infty, 3]$ and is increasing on $[3, +\infty)$.

Example 2. Let us consider the function $f(x) = x^3$. Its derivative $f'(x) = 3x^2$, so $f'(x) > 0$ for $x \neq 0$. We conclude that f is increasing on $(-\infty, 0]$ and on $[0, +\infty)$, so it is in fact increasing everywhere (which confirms what we already know about this function).

Examples

Example 3. Let us consider the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$. Its derivative is

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2).$$

We see that $f'(c) = 0$ for $c = 0, 1, -2$. Let us determine the sign of f' at all the remaining points.

interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 1)$	$(1, +\infty)$
signs of factors x, $(x-1)$, $(x+2)$	$(-)(-)(-)$	$(-)(-)(+)$	$(+)(-)(+)$	$(+)(+)(+)$
sign of f'	$-$	$+$	$-$	$+$

We conclude that f is decreasing on $(-\infty, -2]$ and $[0, 1]$, and is increasing on $[-2, 0]$ and $[1, +\infty)$.

Relative minima and maxima

Suppose f is defined on an open interval containing c. It is said to have a *relative minimum* ("local minimum") at c, if for x sufficiently close to c we have $f(x) \geq f(c)$. Similarly, it is said to have a *relative maximum* ("local maximum") at c, if for x sufficiently close to c we have $f(x) \leq f(c)$. For short, the expression *relative extremum* is also used when referring to points where either a relative minimum or a relative maximum is attained.

Example. The function $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima. In fact, this function attains its minimal value at $x = 0$, so it is not just a relative minimum. The function $f(x) = \cos x$ has relative minima at all odd multiples of π (where it attains the value -1), and relative maxima at all even multiples of π (where it attains the value 1).

Critical points

Theorem. Suppose that f is defined on an open interval containing c, and has a local extremum at c. Then either $f'(c) = 0$ or f is not differentiable at c.
Example. The function $f(x) = |x|$ has a relative minimum at $x = 0$, but is not differentiable at that point.

Points c where f is either not differentiable or has the zero derivative are called *critical points* of f. Among the critical points, the points where $f'(c) = 0$ are called stationary points.

Example. Let us determine the critical points of the function $f(x) = x - \sqrt[3]{x}$. We have $f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}}$, so f' is not defined at $x = 0$, and is zero at $x = \pm\frac{1}{\sqrt{27}}$. The latter two are the stationary points of f.

Local extrema: example

Example. Let us consider the function $f(x) = x^4 - x^3 + 1$ on $[-1, 1]$. Suppose we would like to find all its relative extrema. This function is differentiable everywhere, so "suspicious" points are just the stationary points. To determine them, we compute the derivative:

$$f'(x) = 4x^3 - 3x^2.$$

Points c for which $f'(c) = 0$ are $c = 0$ and $c = 3/4$. How to proceed from here? Let us note that $f'(x) < 0$ for $-1 \leq x < 0$ and $0 < x < 3/4$, and $f'(x) > 0$ for $x > 3/4$. This means that $f(x)$ is decreasing on $[-1, 0]$ and $[0, 3/4]$, and is increasing on $[3/4, 1]$. This in turn means that at $x = 3/4$ a relative minimum is attained, that at points $x = -1$ and $x = 1$ relative maxima are attained, and at the point $x = 0$ we do not have a local extremum at all.

First derivative test

First derivative test for relative extrema. Suppose that f is continuous at its critical point c.

- If $f'(x) > 0$ on some open interval extending left from c, and $f'(x) < 0$ on some open interval extending right from c, then f has a relative maximum at c.
- If $f'(x) < 0$ on some open interval extending left from c, and $f'(x) > 0$ on some open interval extending right from c, then f has a relative minimum at c.
- If $f'(x)$ has the same sign on some open interval extending left from c as it does on some open interval extending right from c, then f does not have a local extremum at c.

Proof of validity. In the first case, $f'(x) > 0$ on some interval (a, c), and $f'(x) < 0$ on some interval (c, b). This means that f is increasing on $[a, c]$ and decreasing on $[c, b]$, from which we easily infer that f has a relative maximum at c. The other cases are similar.

First derivative test: example

Example. Let us analyse the stationary points of the function $f(x) = x - \sqrt[3]{x}$ we considered earlier. We recall that $f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}}$, so for the stationary point $x = -\frac{1}{\sqrt{27}}$, we have $f'(x) > 0$ on an open interval extending left from that point, and $f'(x) < 0$ on an open interval extending right from that point, and for the stationary point $x = \frac{1}{\sqrt{27}}$, we have $f'(x) < 0$ on an open interval extending left from that point, and $f'(x) > 0$ on an open interval extending right from that point.

We conclude that f has a relative maximum at $x = -\frac{1}{\sqrt{27}}$, and a relative minimum at $x = \frac{1}{\sqrt{27}}$.

Second derivative test

The first derivative test is useful, but involves finding the corresponding open intervals where we can analyse the behaviour of the sign of f'. Sometimes a simpler test is available, which just amounts to computing the sign of an individual number.

Second derivative test for relative extrema. Suppose that f is twice differentiable at the point c.

- If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive: the function f may have a relative maximum, relative minimum, or no relative extrema at all at the point c.

Proof of validity. In the first case, $f''(c) = \lim\limits_{x\to c} \frac{f'(x)-f'(c)}{x-c} = \lim\limits_{x\to c} \frac{f'(x)}{x-c}$ is negative, so $f'(x) > 0$ on some open interval extending left from c, and $f'(x) < 0$ on some open interval extending right from c, and the first derivative test applies. The second case is similar. In the third case, the examples $f(x) = x^4$, $f(x) = -x^4$, and $f(x) = x^3$ (at the point $c = 0$ show that "anything can happen".

Second derivative test: example

Example. Let us analyse the stationary points of the function $f(x) = \frac{x}{2} - \sin x$ on $[0, 2\pi]$. We have

$$f'(x) = \frac{1}{2} - \cos x,$$

and

$$f''(x) = \sin x.$$

The points c in $[0, 2\pi]$ where the first derivative vanishes are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$. Substituting into the second derivative, we get

$$f''(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, \quad f''(\frac{5\pi}{3}) = \sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}.$$

We conclude that f has a relative maximum at $\frac{5\pi}{3}$, and a relative minimum at $\frac{\pi}{3}$.