

1S11: CALCULUS FOR STUDENTS IN SCIENCE

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TCD

Lecture 1

PRACTICALITIES

- no lecture notes:
 - ▶ mostly computer presentations plus some blackboard-and-chalk lectures
 - ▶ you're encouraged to take notes and check out the textbook when needed
- there will be “tutorials”: one per week for each half of 1S11, basically an exercise class where you will get some assistance with questions if needed
 - ▶ tutorials help to get prepared to the final exam
 - ▶ working in tutorials improves your mark: the final mark is computed as $0.8F+0.2T$, where F is the final exam mark, T is average of mark from tutorials
- my office is 19.32A Westland Row, access through School of Maths in Hamilton Building, office hours for this module Monday 1pm-2pm or by appointment (e-mail vdots@maths.tcd.ie)
- all available course materials will be posted on the module webpage <http://www.maths.tcd.ie/~vdots/teaching/1S11-1314.html>

INTRODUCTION

- *Bitte vergiß alles, was Du auf der Schule gelernt hast; denn du hast es nicht gelernt.*¹ — Edmund Landau, Grundlagen der Analysis.
- By its very nature, maths is a precise science. It has both good and bad sides...
 - ▶ “good”: once everything is proved, it’s proved once and forever,
 - ▶ “bad”: you have to prove everything!
- In this module, we shall sometimes allow ourselves to “believe” in mathematical statements that we can use for effective computations. However, we shall also prove things, and it is important to keep track of what you can prove, and what you just take for granted.

¹Please forget whatever you’ve been studying at school; for you have not learned it.

CALCULUS

- Calculus is a mathematical way of describing the real world: based on the idea that however complex an object may be globally, it is no different from other objects if you look at it closely, “using a strong microscope” .
- One of the main observations is that in real world there are “independent variables” (coordinates) and “dependent variables” (functions), and so we should develop tools to study mathematical dependencies.
- **Example:** geographic location (3D coordinates) and time are independent variables describing our physical reality, while outside temperature is a dependent variable (you can arrange to be in Barcelona at 8am on October 28, or in London at 4pm on December 13, but once you are there, you cannot choose what the weather is going to be, and the temperature depends both on your location and on the time and season of your stay).

FUNCTIONS

Definition. A (real) variable y is said to be a *function* of a (real) variable x if its dependence on x is such that for every a there is exactly one value of y corresponding to the value $x = a$.

A function can be represented by a table, for example

x	1	1.5	2	2.5	3	3.5	4
y	0	1.25	3	5.25	8	11.25	15

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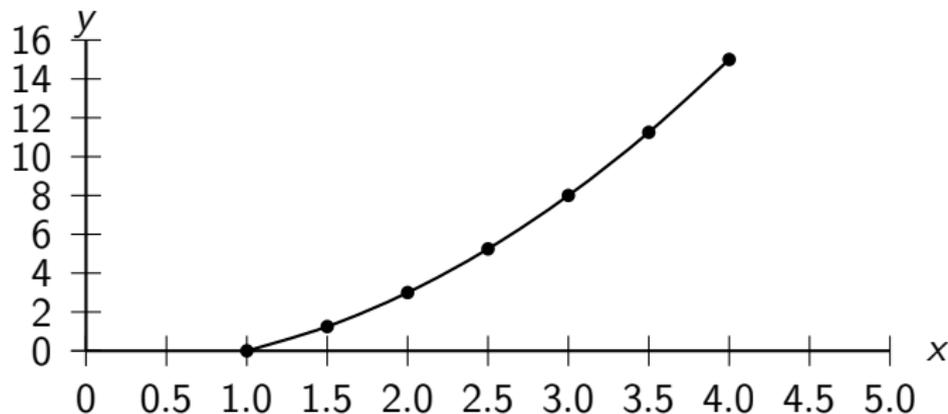
A function can be represented by an algebraic formula, for example

$$y = x^2 - 1$$

FUNCTIONS

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A function can be represented by a graph, for example



FUNCTIONS

Definition. A (real) variable y is said to be a *function* of a (real) variable x if its dependence on x is such that for every a there is exactly one value of y corresponding to the value $x = a$.

A function can be represented verbally, for example “to compute the value of y corresponding to a certain value, one should multiply that value by itself and subtract 1”.

FUNCTIONS

(A slightly better) Definition. A (real valued) *function* f of a (real) variable x is a rule that associates a unique output with each input value of x . That output is denoted by $f(x)$. **Notation:** $x \mapsto f(x)$.

Let us note that f is the name of the rule, so $x \mapsto f(x)$ and $t \mapsto f(t)$ denote exactly the same function f , it is just that we use different names for the independent variable.

FUNCTIONS VIA TABLES

If a function is represented by a table, figuring out what corresponds to what is very straightforward:

x	1	2	3	4	5	6	7
y	4	5	0	1	3	4	-1

FUNCTIONS VIA FORMULAS

If a function is represented by a formula, it is just a matter of a computation to figure out what corresponds to what:

$$f(x) = x^2 - 2x + 3$$

means that

$$f(0) = 0^2 - 2 \cdot 0 + 3 = 3,$$

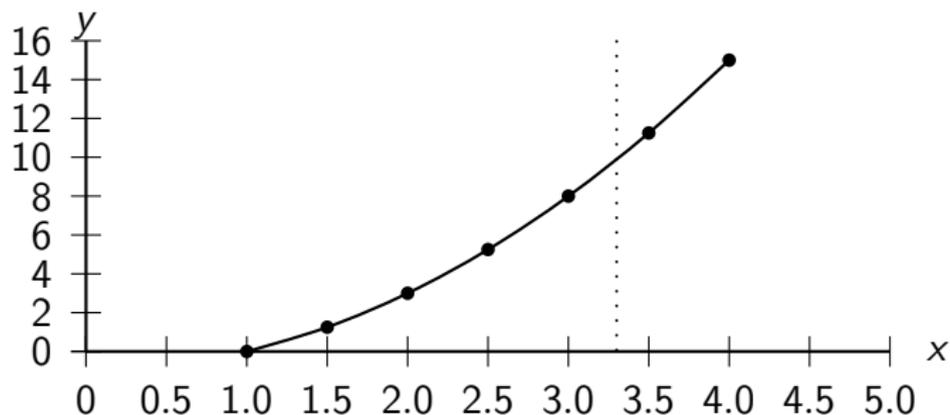
$$f(1) = 1^2 - 2 \cdot 1 + 3 = 2,$$

$$f(-2.5) = (-2.5)^2 - 2 \cdot (-2.5) + 3 = 14.25,$$

$$f(\sqrt{3}) = (\sqrt{3})^2 - 2\sqrt{3} + 3 = 6 - 2\sqrt{3}.$$

FUNCTIONS VIA GRAPHS

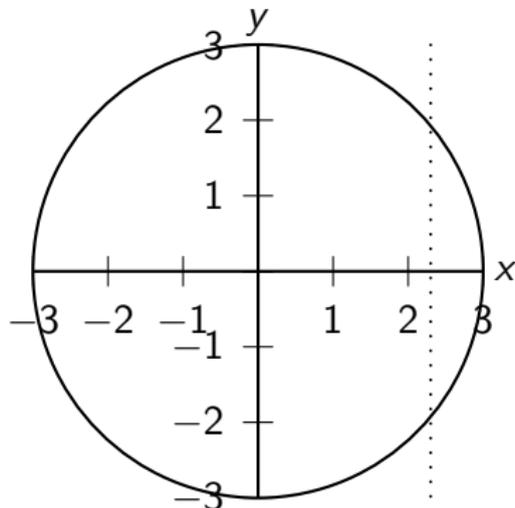
If a function can be represented by a graph, to figure out what corresponds to a particular value of x , one draws a vertical line intersecting the x axis at that value, and looks for where it intersects the graph:



This leads to a useful *vertical line test*: for a curve in the xy -plane to be the graph of some function, no vertical line can meet it in more than one point.

VERTICAL LINE TEST: AN EXAMPLE

Points (x, y) for which $x^2 + y^2 = 9$ form a circle of radius 3, and there are many vertical lines meeting that circle twice:

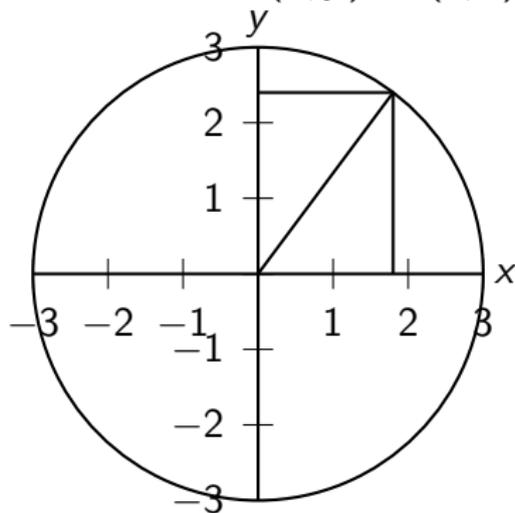


Therefore that circle is not a graph of a function.

Algebraically, $y^2 = 9 - x^2$ means that $y = \pm\sqrt{9 - x^2}$, so the value of y can only be determined up to a sign, and is in no way unique.

BY THE WAY...

Why do points (x, y) for which $x^2 + y^2 = 9$ form a circle of radius 3?
Because $x^2 + y^2$ is the distance from (x, y) to $(0, 0)$ by the Pythagoras Theorem:



SQUARE ROOTS

Recall that every positive real number has two square roots, a positive one and a negative one. By \sqrt{x} we shall denote the positive square root of x , and zero for $x = 0$. A negative number does not admit a real square root at all, since the square of any real number is always nonnegative: $t^2 \geq 0$ for all t .

If t is nonnegative, then $\sqrt{t^2} = t$. However, for negative t this equation does not hold: for example, if $t = -3$, then

$$\sqrt{t^2} = \sqrt{(-3)^2} = \sqrt{9} = 3 \neq t.$$

By inspection, one can show that

$$\sqrt{t^2} = \begin{cases} t, & t \geq 0, \\ -t, & t < 0. \end{cases}$$

ABSOLUTE VALUES

The *absolute value* or *magnitude* of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

In other words, it assigns to a number x that same number x if it is nonnegative, and strips away the minus sign if it is negative. For example, $|3| = 3$, $|-3.5| = 3.5$, $|0| = 0$.

Important properties of absolute values are

$$|-a| = |a|,$$

$$|ab| = |a||b|,$$

$$|a/b| = |a|/|b|, \quad b \neq 0,$$

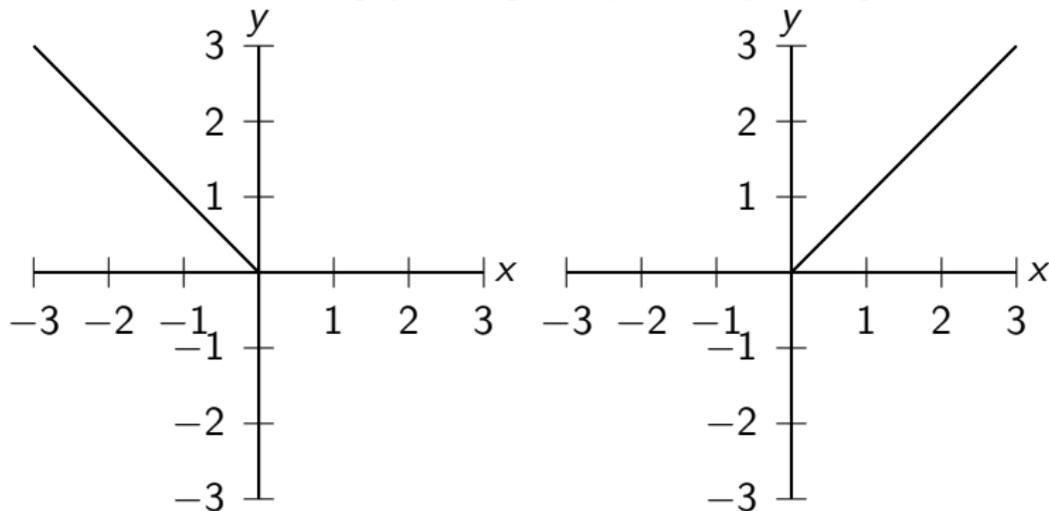
$$|a + b| \leq |a| + |b| \quad (\text{“triangle inequality”}),$$

$$\sqrt{a^2} = |a|.$$

PIECEWISE DEFINED FUNCTIONS

The absolute value is an example of a *piecewise defined function*: it is defined by one formula ($f(x) = x$) for $x \geq 0$ and by another formula ($f(x) = -x$) for $x < 0$.

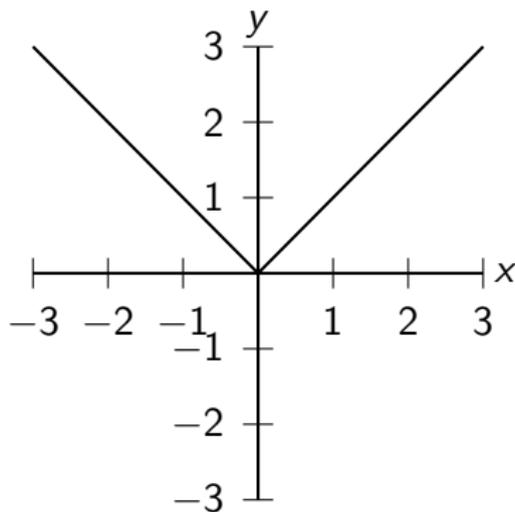
Its graph can be obtained by plotting two parts separately:



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Its graph can be obtained by plotting two parts separately and joining them together:



PIECEWISE DEFINED FUNCTIONS

“Pieces” of the real line on which a piecewise defined function is defined are usually intervals or rays with endpoints included or excluded.

Notation: $[a, b]$ stands for the *closed interval* consisting of all x for which $a \leq x \leq b$, (a, b) stands for the *open interval* consisting of all x for which $a < x < b$, $[a, b)$ stands for the interval consisting of all x for which $a \leq x < b$, and $(a, b]$ stands for the interval consisting of all x for which $a < x \leq b$.

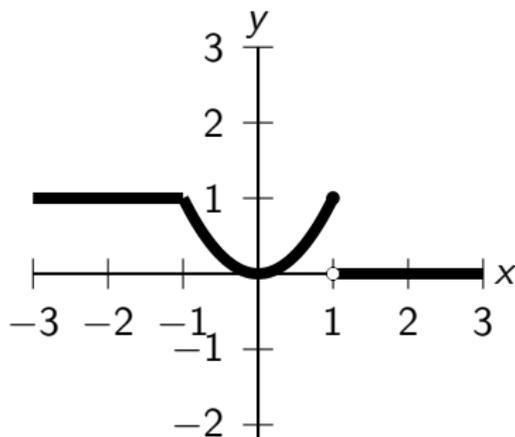
An open end of an interval may assume infinite values, in which case an interval becomes a ray: for example, $[a, +\infty)$ stands for the *closed ray* consisting of all x for which $a \leq x$, and $(-\infty, a)$ stands for the *open ray* consisting of all x for which $x < a$. Of course $(-\infty, +\infty)$ is just a symbol notation for the whole real line.

PIECEWISE DEFINED FUNCTIONS

As an example, let us consider the function

$$f(x) = \begin{cases} 1, & x \leq -1, \\ x^2, & -1 < x \leq 1, \\ 0, & x > 1. \end{cases}$$

It is important to be particularly careful when computing the values and plotting graphs at the endpoints of intervals since the values on the left and on the right may not agree, so it is important which intervals are open and which are closed:



DOMAIN OF A FUNCTION

The square root function $x \mapsto \sqrt{x}$ is an example of a function which is only defined for some values of the independent variable x , in this case for $x \geq 0$. Another example is the inversion $x \mapsto 1/x$, in which case all values of x except for $x = 0$ are allowed.

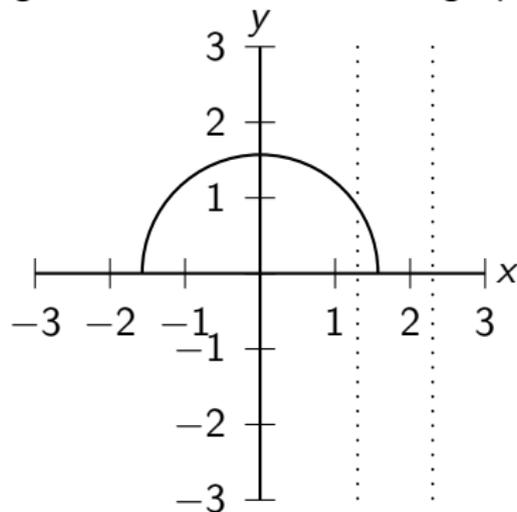
Definition. The *domain* of a function f consists of all values of x for which the value $f(x)$ is defined.

For a function defined by a table, its domain consists of numbers in the first row:

x	1	1.5	2	2.5	3	3.5	4
y	0	1.25	3	5.25	8	11.25	15

DOMAIN OF A FUNCTION

For a function defined by a graph, its domain consists of the values for which the corresponding vertical line meets the graph:



In other words, the domain is the projection of the graph on the x -axis.

NATURAL DOMAIN OF A FUNCTION

With the expressions like \sqrt{x} or $1/x$ we saw how the domain of a function may be restricted by a formula from which we compute values of a function.

Definition. The *natural domain* of a function f defined by a formula consists of all values of x for which $f(x)$ has a well defined real value.

Example 1. The natural domain of $f(x) = x^3$ consists of all real numbers $(-\infty, \infty)$, since for each real x its cube is a well defined real number.