MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 9

Week 11, Michaelmas 2013

1. Find the indefinite integrals of $x^5 - 4x$, $\sqrt[3]{x} + 1/\sqrt[3]{x}$, $x^2 - \cos x$ and $(x^3 - 1)(x + 2)$.

   Solution. In this problem, we only ever need to use the formula $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and the linearity property. Indeed,

$$\int (x^5 - 4x)\,dx = \int x^5\,dx - 4\int x\,dx = \frac{x^6}{6} - 4\frac{x^2}{2} + C = \frac{x^6}{6} - 2x^2 + C,$$

$$\int (\sqrt[3]{x} + 1/\sqrt[3]{x}) = \int (x^{1/3} + x^{-1/3})\,dx = \frac{x^{4/3}}{4/3} + \frac{x^{2/3}}{2/3} + C,$$

$$\int (x^2 - \cos x)\,dx = \frac{x^3}{3} - \sin x + C,$$

$$\int (x^3 - 1)(x + 2)\,dx = \int (x^4 + 2x^3 - x - 2)\,dx = \frac{x^5}{5} + \frac{x^4}{2} - \frac{x^2}{2} - 2x + C.$$

2. Evaluate the following integrals using u-substitution:

$$\int (3x^2 - 5)^{1/3} x\,dx, \qquad \int \cos^2(x)\sin(x)\,dx$$

   Solution. With $u = 3x^2 - 5$ we have $du = 6x\,dx$, so that

$$\int (3x^2 - 5)^{1/3} x\,dx = \frac{1}{6}\int u^{1/3}\,du = \frac{1}{6}\frac{3}{4}u^{4/3} + C = \frac{1}{8}(3x^2 - 5)^{4/3} + C,$$

   and for the second integral we obtain with $u = \sin(x)$ and $du = \cos(x)\,dx$

$$\int \sin^2(x)\cos(x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C.$$

3. Evaluate

$$\int \frac{1}{1 - \sin x}\,dx.$$

   Solution. We first transform the integrand:

$$\frac{1}{1 - \sin x} = \frac{1}{1 - \sin x}\frac{1 + \sin x}{1 + \sin x} = \frac{1 + \sin x}{1 - \sin^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}. \qquad (1)$$

   The first summand gives an integral from the integral table, and the second one is solved by u-substitution, as $\sin x\,dx = -d(\cos x)$, so

$$\int \frac{\sin x}{\cos^2 x}\,dx = -\int \frac{d(\cos x)}{\cos^2 x} = \frac{1}{\cos x} + C,$$

and

$$\int \frac{1}{1-\sin x}\,dx = \tan x + \frac{1}{\cos x} + C$$

4. Use (repeated) integration by parts to evaluate the integral $\int (x^3-1)e^x\,dx$.

   Solution. We have

$$\begin{aligned}
&\int (x^3-1)e^x\,dx = \\
&= \int (x^3-1)d(e^x) = (x^3-1)e^x - \int e^x\,d(x^3-1) = (x^3-1)e^x - \int e^x(3x^2)\,dx = \\
&= (x^3-1)e^x - \int 3x^2\,de^x = (x^3-1)e^x - 3x^2e^x + \int e^x\,d(3x^2) = \\
&= (x^3-1)e^x - 3x^2e^x + \int e^x(6x)\,dx = (x^3-1)e^x - 3x^2e^x + \int 6x\,d(e^x) = \\
&= (x^3-1)e^x - 3x^2e^x + 6xe^x - \int e^x\,d(6x) = (x^3-1)e^x - 3x^2e^x + 6xe^x - 6\int e^x\,dx = \\
&= (x^3-1)e^x - 3x^2e^x + 6xe^x - 6e^x + C = \\
&= (x^3-3x^2+6x-7)e^x + C.
\end{aligned}$$