MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 6

Week 8, Michaelmas 2013

1. Differentiate
$$f(x) = \frac{1}{x^2} + 4x^{3/2}, \tag{1}$$
and
$$f(x) = x + \frac{1}{x} + \frac{3}{4}x^{4/3}. \tag{2}$$

Solution. We apply the rule for differentiating power functions, and the fact that derivatives agree with sums and scalar factors:
$$\left(\frac{1}{x^2} + 4x^{3/2}\right)' = \left(x^{-2} + 4x^{3/2}\right)' = -2x^{-2-1} + 4\cdot\frac{3}{2}x^{3/2-1} = -\frac{2}{x^3} + 6\sqrt{x},$$
and
$$\left(x + \frac{1}{x} + \frac{3}{4}x^{4/3}\right)' = \left(x^1 + x^{-1} + \frac{3}{4}x^{4/3}\right)' =$$
$$= 1\cdot x^0 + (-1)x^{-1-1} + \frac{3}{4}\cdot\frac{4}{3}x^{4/3-1} = 1 - \frac{1}{x^2} + \sqrt[3]{x}.$$

2. Show that for the curves $y = 1/x$ and $y = 1/(2-x)$, their tangent lines at the point where those two curves meet are perpendicular to one another.

Solution. Let us find the intersection point for these curves. If $1/x = y = 1/(2-x)$, then $x = 2 - x$, so $2x = 2$, $x = 1$. In this case, $y = 1/x = 1$ also. The slope of the tangent line to the first curve is
$$\left.\left(\frac{1}{x}\right)'\right|_{x=1} = \left.\left(-\frac{1}{x^2}\right)\right|_{x=1} = -1,$$
and the tangent line is $y - 1 = -(x-1)$, that is $y = -x + 2$. The slope of the tangent line to the second curve is
$$\left.\left(\frac{1}{2-x}\right)'\right|_{x=1} = \left.\left(-(-1)\frac{1}{(2-x)^2}\right)\right|_{x=1} = 1,$$
and the tangent line is $y - 1 = (x-1)$, that is $y = x$. These two lines are manifestly perpendicular to one another.

3. Using the chain rule and rules for derivatives of trigonometric functions, compute the derivative function of $f(x) = \sin^2 x + \cos^2 x$. Explain why your answer is consistent with the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

   Solution. We have

   $$\left(\sin^2 x + \cos^2 x\right)' = 2\sin x(\sin x)' + 2\cos x(\cos x)' = 2\sin x\cos x + 2\cos x(-\sin x) = 0.$$

   This agrees with the fact that $\sin^2 x + \cos^2 x = 1$, since the derivative of a constant is zero.

4. Differentiate

   $$f(x) = x\cos x + \sqrt{1-x^2}, \tag{3}$$

   and

   $$f(x) = \frac{\sin x}{x + 3\sqrt[3]{x}}. \tag{4}$$

   Solution. Because of the product rule and the chain rule, we have

   $$\left(x\cos x + \sqrt{1-x^2}\right)' = (x)'\cos x + x(\cos x)' + \frac{1}{2\sqrt{1-x^2}}(1-x^2)' = $$
   $$= \cos x - x\sin x - \frac{x}{\sqrt{1-x^2}}.$$

   Because of the rule of differentiating quotients, we have

   $$\left(\frac{\sin x}{x + 3\sqrt[3]{x}}\right)' = \frac{\cos x(x + 3\sqrt[3]{x}) - \sin x\left(1 + 3\frac{1}{3\sqrt[3]{x^2}}\right)}{(x + 3\sqrt[3]{x})^2} =$$
   $$= \frac{\cos x(x + 3\sqrt[3]{x}) - \sin x\left(1 + \frac{1}{\sqrt[3]{x^2}}\right)}{(x + 3\sqrt[3]{x})^2}.$$

5. Let $f(x) = \sin x$, with the domain $(-\pi/2, \pi/2)$, where this function is increasing and therefore is invertible. Apply the chain rule to the equation

   $$f\left(f^{-1}(x)\right) = x \tag{5}$$

   to show that the derivative of $f^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. (*Hint*: you may need the identity $\cos^2 x + \sin^2 x = 1$).

   Solution. Applying the chain rule to $f\left(f^{-1}(x)\right) = x$, we get

   $$f'(f^{-1}(x))(f^{-1}(x))' = 1,$$

   so

   $$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(f^{-1}(x))} = \frac{1}{\sqrt{1-(\sin(f^{-1}(x)))^2}} = \frac{1}{\sqrt{1-x^2}},$$

   since on $(-\pi/2, \pi/2)$ we have $\cos t > 0$ and hence $\cos t = \sqrt{1-\sin^2 t}$, and since $\sin(f^{-1}(x)) = f(f^{-1}(x)) = x$.