MA1S11 (Dotsenko) Solutions to Tutorial/Exercise Sheet 3

Week 4, Michaelmas 2013

1. Complete the sentences and explain your answer.

   - "If $\lim_{x\to a} f(x) = 3$, then $\lim_{x\to a} \sqrt{6+f(x)}$ is…".
   - "If $\lim_{y\to a} g(y) = 1$, then $\lim_{y\to a} \frac{g(y)}{3-g(y)}$ is…".

   Solution. Since $\lim_{x\to a} f(x) = 3$, we have $\lim_{x\to a}(6+f(x)) = 9$, and $\lim_{x\to a} \sqrt{6+f(x)} = 3$. Also, since $\lim_{y\to a} g(y) = 1$, we have $\lim_{y\to a}(3-g(y)) = 2$, so $\lim_{y\to a} \frac{g(y)}{3-g(y)} = \frac{\lim_{y\to a} g(y)}{\lim_{y\to a}(3-g(y))} = \frac{1}{2}$.

2. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim_{x\to 2^+} \frac{x^2-2x+1}{x^2-3x+2}$.
   - $\lim_{x\to 2^-} \frac{x^2-2x+1}{x^2-3x+2}$.
   - $\lim_{x\to 2} \frac{x^2-2x+1}{x^2-3x+2}$.

   Solution. Let us note that $\frac{x^2-2x+1}{x^2-3x+2} = \frac{(x-1)^2}{(x-1)(x-2)} = \frac{x-1}{x-2}$. Therefore, the first limit is $+\infty$ (the numerator is positive if x is close to 2, the denominator is positive on the right from 2, and increases without bound), the second limit is $-\infty$ (the numerator is positive if x is close to 2, the denominator is negative on the left from 2, and decreases without bound), and the third limit does not exist, since the limits on the left and on the right do not agree.

3. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim_{x\to 1^+} \frac{x^2-2x+1}{x^2-3x+2}$.
   - $\lim_{x\to 1^-} \frac{x^2-2x+1}{x^2-3x+2}$.
   - $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-3x+2}$.

   Solution. Let us note that $\frac{x^2-2x+1}{x^2-3x+2} = \frac{(x-1)^2}{(x-1)(x-2)} = \frac{x-1}{x-2}$. Therefore, all the limits exist and are equal to 0, since $\frac{x-1}{x-2}$ is defined at $x=1$ and is equal to 0.

4. Which of the following limits exist (as finite or infinite limits)? Explain your answer and compute them.

   - $\lim_{x\to 1} \frac{1}{|x-1|}$.
   - $\lim_{x\to 1} \frac{x^2-3x+2}{(x-1)^3}$.

   Solution. The first limit is $+\infty$ (the numerator is positive, the denominator is positive for $x \neq 1$, and increases without bound). To compute the second limit, we observe that $\frac{x^2-3x+2}{(x-1)^3} = \frac{(x-1)(x-2)}{(x-1)^3} = \frac{x-2}{(x-1)^2}$, so the limit as $x \to 1$ is $-\infty$ (the numerator is negative if x is close to 1, the denominator is positive for $x \neq 1$ and increases without bound).

5. Consider the function $f(x) = x^2 + x$. Find the equation for the line passing through the points $(-1, 0)$ and $(x, f(x))$ on the graph of that function, and determine the equation of the tangent line to the graph at $(-1, 0)$ by computing the limiting position of that line as $x \to -1$.

   Solution. By point-slope formula, any line passing through $(-1, 0)$ has the equation $y = m(x+1)$ for some m. If the point $(x, f(x))$ is on that line, then $f(x) = m(x+1)$, so
   $$m = \frac{f(x)}{x+1} = \frac{x^2+x}{x+1} = \frac{x(x+1)}{x+1} = x$$
   for $x =\neq -1$. Therefore, as $x \to -1$, the limit of m is the limit of x, that is -1, so the equation of the tangent line is $y = -x - 1$.