Hierarchies of identities for differential operators and moduli spaces of curves

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joint with Sergey Shadrin (Amsterdam) and Bruno Vallette (Nice)

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**Motivation**

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(signs needed for elements of degree $\neq 0$), which expresses the fact that $\Delta$ is a “differential operator of order at most 2”. Often presented together with an auxiliary operation

$$[a, b] = \Delta(ab) - \Delta(a)b - a\Delta(b),$$

but we shall not need it.
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BV-algebras, among other things, lead to connections between:

- mathematical formulations of field theory (e.g. 2d TFT / CFT / CohFT)
- algebraic geometry (Calabi–Yau manifolds, mirror symmetry conjecture, moduli spaces of curves)
- homotopical algebra in general

Getzler '94, Barannikov & Kontsevich '97, Losev & Shadrin '05, and many others

One of recent results (Drummond-Cole & Vallette '09) reveals a relationship between the homotopy theory for BV-algebras and moduli spaces of curves with marked points.
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An early approach to homotopy theory for BV-algebras (Kravchenko ’99), “commutative homotopy BV-algebras”: the associativity is still strict, but \( \Delta_2 \) only vanishes up to homotopy.

More precisely: algebra \((V, d)\) with operators \(\Delta_1 = d\), \(\Delta_2 = \Delta\), \(\Delta_3, \ldots\), where \(\Delta_k\) is of degree \(2k - 3\), and

\[
\sum_{i+j=n} \Delta_i \Delta_j = 0.
\]

(For instance, \(d \Delta_3 + \Delta_3 d + \Delta_2 = 0\), so \(\Delta_3\) provides a homotopy for the relation \(\Delta_2 = 0\).)

Additional requirement: each \(\Delta_k\) is a differential operator of order at most \(k\). (Needed if we do not want any further higher structures!)
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Additional requirement: each $\Delta_k$ is a differential operator of order at most $k$. (Needed if we do not want any further higher structures!)
Definition (Grothendieck, EGA IV). Let $A$ be an associative commutative algebra. A differential operator of order at most 0 is an operator of the form $f \cdot (-): g \mapsto f \cdot g$ for some $f \in A$.

Furthermore, a linear operator $D: A \to A$ is a differential operator of order at most $p$ if $[D, f \cdot (-)]$ is a differential operator of order at most $p - 1$ for every $f \in A$. 
**Definition (Koszul, 1980s).** Let $A$ be an associative commutative algebra. The hierarchy of brackets $\langle -, -, \ldots, - \rangle^D_p : A^\otimes p \to A$ associated to a linear operator $D : A \to A$ is defined recursively by $\langle f \rangle^D_1 := D(f)$ and

\[
\langle f_1, \ldots, f_{p-1}, f_p, f_{p+1} \rangle^D_{p+1} = \langle f_1, \ldots, f_{p-1}, f_pf_{p+1} \rangle^D_p - \langle f_1, \ldots, f_{p-1}, f_p \rangle^D_pf_{p+1} - f_p\langle f_1, \ldots, f_{p-1}, f_{p+1} \rangle^D_p.
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An operator $D$ is a **differential operator order at most $p$** if the bracket $\langle -, -, \ldots, - \rangle^D_{p+1}$ is identically equal to zero.
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For algebras with unit, the two definitions are equivalent if we restrict ourselves to operators that annihilate the unit. In this talk, we stick to the second definition.
A hierarchy of formulas

For a differential operator $D$ of order at most $p$, any $n \geq p + 1$, and integers $d_0, \ldots, d_n \geq 0$ with $p + d_0 + d_1 + \cdots + d_n = n - 1$,}

\[
\left(\begin{array}{c} n-2 \\ d_0 + p - 1, d_1, \ldots, d_n \end{array}\right)D(f_1 f_2 \cdots f_n) + \\
+ \sum_{i+j=p-2} (-1)^{i+1} \left(\begin{array}{c} |I| - 2 \\ i, d_{i_1}, \ldots, d_{i_r} \end{array}\right) \left(\begin{array}{c} |J| - 1 \\ j, d_0, d_{j_1}, \ldots, d_{j_s} \end{array}\right) D(f_I f_J) + \\
+ (-1)^p \sum_{m=1}^n \left(\begin{array}{c} n-2 \\ d_0, \ldots, d_m + p - 1, \ldots, d_n \end{array}\right) f_1 \cdots D(f_m) \cdots f_n = 0,
\]

where in the last sum the summation is also over all $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_s\}$ with $r \geq 2$, $I \sqcup J = \{1, \ldots, n\}$, and $i + d_{i_1} + \ldots + d_{i_r} = |I| - 2$, $j + d_0 + d_{j_1} + \ldots + d_{j_s} = |J| - 1$. 
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- For a differential operator $D$ of order at most 1, the whole hierarchy reduces to

\[ D(f_1 f_2 \cdots f_n) = \sum_{m=1}^{n} D(f_m)f_1 \cdots \hat{f}_m \cdots f_n, \]

but also includes infinitely many other ones.
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- For a differential operator $D$ of order at most 2, the hierarchy includes

$$D(f_1 f_2 \cdots f_n) = \sum_{1 \leq i < j \leq n} D(f_j f_j) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n - (n - 2) \sum_{m=1}^{n} D(f_m) f_1 \cdots \hat{f}_m \cdots f_n,$$

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  - (also can include projections $\pi: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting the last marked point, but we shall ignore them most of the time).
Modular operads

Generally, an algebraic modular operad is something modelled on the collection of modular spaces and maps between them, that is a collection of graded vector spaces $P_{g,n}$ with appropriate maps $\sigma, \rho$.

Example: the "modular endomorphism operad" of a vector space with a scalar product: $(\text{End } V)^{g,n} = V^\otimes n$, with $\sigma$ and $\rho$ given by contractions.
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An *algebra* over a modular operad $P$ is a vector space $V$ with a scalar product together with a morphism of modular operads $P \to \text{End}_V$. 
Cohomological field theories

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A cohomological field theory is an algebra over the modular operad \( \{ H_{\bullet}(\overline{M}_{g,n}) \} \).

In other words, a cohomological field theory on a vector space \( V \) is a collection of (co)homology classes

\[
\alpha_{g,n} \in \text{Hom}(H_{\bullet}(\overline{M}_{g,n}), V^{\otimes n}) \approx H^{\bullet}(\overline{M}_{g,n}) \otimes V^{\otimes n}
\]

behaving well with respect to pushforwards via \( \sigma \) and \( \rho \).
A cohomological field theory is said to be a *topological field theory* if all the classes $\alpha_{g,n}$ are of cohomological degree 0. In this case, they all are determined by $\alpha_{0,3}$, which, viewed as an element of $H^\bullet(\overline{M}_{0,3}) \otimes V^\otimes 3 \simeq \text{Hom}(V^\otimes 2, V)$, should define a commutative associative product on $V$. Including projections $\pi$ amounts to considering algebras with unit. If we are only interested in the genus 0 part of a CohFT / TFT, we can “eliminate” the scalar product, using it to identify $V^\otimes (n+1)$ with $\text{Hom}(V^\otimes n, V)$. In this case $\alpha_{0,n+1}$ becomes the $(n-1)$-fold iterated commutative product on $V$. 
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If we are only interested in the genus 0 part of a CohFT / TFT, we can “eliminate” the scalar product, using it to identify $V^\otimes(n+1)$ with $\text{Hom}(V^\otimes n, V)$. In this case $\alpha_{0,n+1}$ becomes the $(n - 1)$-fold iterated commutative product on $V$. 
Let $V$ be a vector space with a scalar product $\eta$. The space of Laurent series with coefficients in $V$ has a symplectic structure

$$\langle v \otimes f(z), w \otimes g(z) \rangle = \eta(v, w) \text{Res}(f(-z)g(z)).$$
Givental group action on CohFT’s

Let $V$ be a vector space with a scalar product $\eta$. The space of Laurent series with coefficients in $V$ has a symplectic structure

$$\langle v \otimes f(z), w \otimes g(z) \rangle = \eta(v, w) \text{Res}(f(-z)g(z)).$$

The Givental Lie algebra is the Lie algebra of the “upper triangular subgroup” of the group of symplectomorphisms of that structure. It consists of all series $r_1z + r_2z^2 + \ldots$, where $r_l \in \text{End}(V)$ is symmetric for odd $l$ and skew-symmetric for even $l$ (with respect to the scalar product $\eta$).
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An action of the Givental Lie algebra on CohFT’s is defined using “tautological classes” of moduli spaces. Main characters: tautological classes $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n})$, the Chern classes of tautological line bundles $\mathbb{L}_1, \ldots, \mathbb{L}_n$ whose fibres are tangent lines at respective marked points.
Givental group action on CohFT’s

The Givental Lie algebra action on cohomological field theories takes the system of classes \( \alpha_{g,n} \in H^\bullet(\mathcal{M}_{g,n}) \otimes V^\otimes n \) to the system of classes \( (r_l z^l.\alpha)_{g,n} \in H^\bullet(\mathcal{M}_{g,n}) \otimes V^\otimes n \) given by the formula

\[
(r_l z^l.\alpha)_{g,n} := \sum_{m=1}^{n} (\alpha_{g,n} \cdot \psi_m^l) \circ_m r_l + \\
\frac{1}{2} \left( \sum_{i=0}^{l-1} (-1)^{i+1} \left( \sigma_*(\alpha_{g-1,n+2} \cdot \psi_{n+1}^i \psi_{n+2}^{l-1-i}), \eta^{-1} r_l \right) + \\
\sum_{i+j=l-1} (-1)^{j+1} \left( \rho_*(\alpha_{g_1,|I|+1} \cdot \psi_{|I|+1}^i \otimes \alpha_{g_2,|J|+1} \cdot \psi_{|J|+1}^j), \eta^{-1} r_l \right) \right),
\]

where the last sum is over all partitions \( I \sqcup J = \{1, \ldots, n\} \) and \( g_1 + g_2 = g \); the maps \( \sigma \) and \( \rho \) identify the points labelled \( n+1 \) and \( n+2 \) in the second sum, and the points \( |I|+1 \) on \( M_{g_1,|I|+1} \) and \( |J|+1 \) on \( M_{g_2,|J|+1} \) in the third sum.
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The Givental Lie algebra action on cohomological field theories takes the system of classes $\alpha_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes V^\otimes n$ to the system of classes $(\tilde{r}_l z^l . \alpha)_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes V^\otimes n$ given by the formula

$$(\tilde{r}_l z^l . \alpha)_{g,n} := \sum_{m=1}^n (\alpha_{g,n} \cdot \psi_m^l) \circ m r_l +$$

$$+ \frac{1}{2} \left( \sum_{i=0}^{l-1} (-1)^{i+1} \left( \sigma_*(\alpha_{g-1,n+2} \cdot \psi_{n+1}^i \psi_{n+2}^{l-1-i}), \eta^{-1} r_l \right) +$$

$$+ \sum_{i+j=l-1} (-1)^{j+1} \left( \rho_*(\alpha_{g_1,|I|+1} \cdot \psi_{|I|+1}^i \otimes \alpha_{g_2,|J|+1} \cdot \psi_{|J|+1}^j), \eta^{-1} r_l \right) \right).$$

The last sum is over all partitions $I \sqcup J = \{1, \ldots, n\}$ and $g_1 + g_2 = g$; the maps $\sigma$ and $\rho$ identify the points labelled $n + 1$ and $n + 2$ in the second sum, and the points $|I| + 1$ on $\overline{\mathcal{M}}_{g_1+1,|I|+1}$ and $|J| + 1$ on $\overline{\mathcal{M}}_{g_2+1,|J|+1}$ in the third sum.
Givental group action on CohFT's

Proposition (Kazarian, Teleman)

The classes

\[ \tilde{\alpha}_{g,n} := \left( \exp \left( \sum_{l=1}^{\infty} \tilde{r}_l \tilde{z}^l \right) \alpha \right)_{g,n} \]

are well-defined cohomology classes with the values in the tensor powers of V that satisfy CohFT constraints; thus, the Givental group acts on cohomological field theories.
Theorem (D.-Shadrin–Vallette, 2011)

Let $A = \{\alpha_{0,n}\}_{n \geq 3}$ be a genus 0 topological field theory on a vector space $V$, making it into a commutative associative algebra. The Lie algebra of the stabiliser of $A$ is spanned by all elements

$$\sum_{p \geq 2} D_p z^{p-1}$$

for which $D_p$ is a differential operator of order at most $p$ on $V$. 

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Sketch of a proof: first, note that for a TFT $A$, all contributions $\hat{D}_p z^{p-1} A$ live in different cohomological degrees, so we may explore these conditions separately.
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$$\int_{\mathcal{M}_{0,n+1}} D_p z^{p-1} \cdot \alpha_n \cdot \prod_{m=0}^{n} \psi^{d_m}_m.$$
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**Sketch of a proof:** first, note that for a TFT $A$, all contributions $\hat{D}_p z^{p-1} A$ live in different cohomological degrees, so we may explore these conditions separately. Second, we get rid of cohomology classes: for a sequence of integers $d_0, \ldots, d_n \geq 0$ with $l - 1 + d_0 + d_1 + \cdots + d_n = n - 2$, we compute

$$\int_{\mathcal{M}_{0,n+1}} D_p z^{p-1}.\alpha_n \cdot \prod_{m=0}^n \psi^{d_m}_m.$$ 

In fact, this can be rewritten as

$$\langle \tau_{d_0+p-1} \tau_{d_1} \cdots \tau_{d_n} \rangle_0 D_p (f_1 f_2 \cdots f_n) +$$

$$+ \sum_{i+j=p-2} (-1)^{j+1} \langle \tau_i \tau_{d_{i_1}} \cdots \tau_{d_{i_r}} \rangle_0 \langle \tau_{d_0} \tau_j \tau_{d_{j_1}} \cdots \tau_{d_{j_r}} \rangle_0 D_p (f_I) f_J +$$

$$+ (-1)^p \sum_{m=1}^n \langle \tau_{d_0} \tau_{d_1} \cdots \tau_{d_{m+p-1}} \cdots \tau_{d_n} \rangle_0 f_1 \cdots D_p (f_m) \cdots f_n.$$
Givental stabilisers of TFT’s, genus 0

On the previous slide, $\langle \tau_{d_0} \tau_{d_1} \cdots \tau_{d_n} \rangle_0$ are the correlators of our TFT (integrals of products of $\psi$-classes).
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It is known that

$$\langle \tau_{d_0} \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \begin{cases} \binom{n-2}{d_0, \ldots, d_n} & \text{if } d_0 + \cdots + d_n = n - 2, \\ 0 & \text{otherwise.} \end{cases}$$
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0 & \text{otherwise}.
\end{cases}
\]

[Aha! That's where the all those coefficients in

\[
\left( \begin{array}{c} n-2 \\ d_0 + p - 1, d_1, \ldots, d_n \end{array} \right) D(f_1 f_2 \cdots f_n) + \\
+ \sum_{i+j=p-2} (-1)^{j+1} \binom{|I| - 2}{i, d_{i_1}, \ldots, d_{i_r}} \binom{|J| - 1}{j, d_0, d_{j_1}, \ldots, d_{j_s}} D(f_i) f_J + \\
+ (-1)^p \sum_{m=1}^{n} \binom{n-2}{d_0, \ldots, d_m + p - 1, \ldots, d_n} f_1 \cdots D(f_m) \cdots f_n,
\]

are coming from!]

Givental stabilisers of TFT’s, genus 0

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0 & \text{otherwise.}
\end{cases}
\]

Also, for any \( i, j \) we have a topological recursion relation

\[
\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_i+1} \cdots \tau_{d_n} \rangle_0 + \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_j+1} \cdots \tau_{d_n} \rangle_0 = \sum_{i \in I, j \in J} \langle \tau_{d_{i_1}} \cdots \tau_{d_{i_r}} \tau_0 \rangle_0 \langle \tau_0 \tau_{d_{j_1}} \cdots \tau_{d_{j_s}} \rangle_0,
\]

hinting that we might be able to prove the theorem by a clever induction.
Givental stabilisers of TFT’s, genus 0

The actual proof is just a little bit more tricky. First, there is a series of identities generalising

\[ D(f_1 f_2 \cdots f_n) = \sum_{1 \leq i < j \leq n} D(f_{ij}) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n - (n - 2) \sum_{m=1}^{n} D(f_m) f_1 \cdots \hat{f}_m \cdots f_n, \]

for operators of order at most 2.
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for operators of order at most 2. Namely, for an operator of order at most \( p \) and for any \( n \geq p + 1 \), we have

\[ D(f_1 f_2 \cdots f_n) = \sum_{I \sqcup J = \{1, \ldots, n\}, \ \ 1 \leq |I| \leq p} (-1)^{p-|I|} \binom{n-1-|I|}{p-|I|} D(f_I) f_J. \]
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\]

\[
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(This identity is the identity for \( d_0 = n - p - 1 \) and \( d_1 = \cdots = d_n = 0 \) above.)
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The actual proof is just a little bit more tricky. First, there is a series of identities generalising

$$D(f_1 f_2 \cdots f_n) = \sum_{1 \leq i < j \leq n} D(f_j f_j) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n -$$

$$- (n - 2) \sum_{m=1}^n D(f_m) f_1 \cdots \hat{f}_m \cdots f_n,$$

for operators of order at most 2. Namely, for an operator of order at most $p$ and for any $n \geq p + 1$, we have

$$D(f_1 f_2 \cdots f_n) = \sum_{|l| \cup |J| = \{1, \ldots, n\}, \quad 1 \leq |l| \leq p} (-1)^{p-|l|} \left( \binom{n-1-|l|}{p-|l|} \right) D(f_l) f_J.$$

(This identity is the identity for $d_0 = n - p - 1$ and $d_1 = \cdots = d_n = 0$ above.) All other identities follow by means of topological recursion relations.
Givental stabilisers of TFT’s, genus 1

If we want to incorporate the genus 1 information, we can “eliminate” the scalar product, using it to identify $V^\otimes(n+1)$ with $\text{Hom}(V^\otimes n, V)$ in the genus 0 part, and $V^\otimes n$ with $\text{Hom}(V^\otimes n, \mathbb{C})$ in the genus 1 part. Main characters: the product $\alpha_{0,3}$ and the “trace” $\alpha_{1,1}$.
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For algebras with trace, the relevant properties of differential operators appear to be Getzler’s 1/12-axiom for order 2:

$$\text{tr}(D_2(f \cdot (-))) = \frac{1}{12} \text{tr}(D_2(f) \cdot (-)),$$
Givental stabilisers of TFT’s, genus 1

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For algebras with trace, the relevant properties of differential operators appear to be Getzler’s 1/12-axiom for order 2:

$$\text{tr}(D_2(f \cdot (-))) = \frac{1}{12} \text{tr}(D_2(f) \cdot (-)),$$

and a (new) property of “being compatible with the trace” for order 3 and higher:

$$\text{tr}(\langle f_1, f_2, \ldots, f_{p-1} \rangle_{p-1}^{D_p} \cdot (-)) = 0.$$
Theorem (D.–Shadrin–Vallette, 2011)

Let $A = \{\alpha_{0,n}\}_{n \geq 3} \cup \{\alpha_{1,n}\}_{n \geq 1}$ be a genus 0 and 1 topological field theory on a vector space $V$, making it into a commutative associative algebra with a trace. The Lie algebra of the stabiliser of $A$ only contains elements

$$\sum_{p \geq 2} D_p z^{p-1}$$

for which $D_p$ is a differential operator of order at most $p$ on $V$ that satisfies Getzler’s $1/12$-axiom for $p = 2$ and is compatible with the trace for $p \geq 3$. 

Givental stabilisers of TFT’s, genus 1
**Givental stabilisers of TFT’s, genus 1**

**Theorem (D.–Shadrin–Vallette, 2011)**

Let $A = \{\alpha_{0,n}\}_{n \geq 3} \cup \{\alpha_{1,n}\}_{n \geq 1}$ be a genus 0 and 1 topological field theory on a vector space $V$, making it into a commutative associative algebra with a trace. The Lie algebra of the stabiliser of $A$ only contains elements

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for which $D_p$ is a differential operator of order at most $p$ on $V$ that satisfies Getzler’s $1/12$-axiom for $p = 2$ and is compatible with the trace for $p \geq 3$. Under the Gorenstein conjecture for genus 1, the Lie algebra of the stabiliser is precisely the linear span of such elements.
Thank you for your patience!