COMPATIBLE ASSOCIATIVE PRODUCTS AND TREES

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arXiv:0809.1773

Glasgow, April 23, 2010
Poisson manifolds

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Properties:

- anti-symmetry: $\{f, g\} = -\{g, f\}$,
- linearity: $\{f, ag_1 + bg_2\} = a\{f, g_1\} + b\{f, g_2\}$ for scalars $a, b$,
- Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,
- Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$. 
Two examples

Example 1 (Kostant–Kirillov Poisson bracket): Let \( g \) be a Lie algebra, then \( g^* \) is a Poisson manifold: the space of linear functions on \( g^* \) is \( g \), so we have a Lie bracket

\[
\{g_1, g_2\}(\xi) = \xi([g_1, g_2]),
\]

which we then extend by Leibniz rule.

Example 2 (constant bracket): In the previous example, fix \( \gamma \in \Lambda^2 g^* \), and let

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which we then extend by Leibniz rule. For instance, if \( g \) is the 2-dimensional solvable Lie algebra, \( g = \text{span} \{p, q| [p, q] = q\} \), we have the Kostant–Kirillov bracket \( \{p, q\} = q \), and a constant bracket \( \{p, q\} = 1 \).
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(AUTONOMOUS) HAMILTONIAN FORMALISM

Let $M$ be a Poisson manifold. To every function $f \in C^\infty(M)$, we relate the corresponding Hamiltonian vector field $X_f \in \Gamma(TM)$ satisfying

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$H \in C^\infty(M)$ a function (energy, Hamiltonian of the system). Hamilton evolution equation associated to $H$:

$$\frac{df}{dt} = X_H(f) = \{H, f\},$$

where $f \in C^\infty(M)$. 

Example: $g$ from the previous slide, $\{p, q\} = 1$:

$$\frac{dp}{dt} = \{-H, p\} = -\frac{\partial H}{\partial q},$$

$$\frac{dq}{dt} = \{-H, q\} = \frac{\partial H}{\partial p}.$$
**Autonomous Hamiltonian formalism**

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A “system” (Poisson manifold) is said to be (Liouville) integrable if it admits a maximal independent involutive set of functions \(\{F_1, \ldots, F_{n-r}\}\). [Here $n = \dim M$, and $2r$ is the rank of the Poisson structure at generic point.]
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It can be shown that the coordinates \( t_i \) can be obtained by algebraic operations, inverting functions and integration. Thus, \textit{integrable systems are solvable in quadratures}, hence the name.
Bi-Hamiltonian systems — 1

Assume that $M$ has two Poisson brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$. These brackets are said to be compatible if $a\{\cdot, \cdot\}_1 + b\{\cdot, \cdot\}_2$ is a Poisson bracket for any choice of scalars $a, b$.

Such a manifold $M$ is called *bi-Hamiltonian*, and every vector field on $M$ which is Hamiltonian with respect to both structures — a *bi-Hamiltonian vector field*.
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Let $X$ be a bi-Hamiltonian vector field, that is $X = \{F_1, \cdot\}_1 = \{F_2, \cdot\}_2$. Then $F_1$ and $F_2$ are in involution with respect to both Poisson structures:

$$\{F_1, F_2\}_1 = \{F_2, F_2\}_2 = 0,$$
$$\{F_1, F_2\}_2 = -\{F_2, F_1\}_2 = -\{F_1, F_1\}_1 = 0.$$
**Lenard recursion formula:** Generalizing the previous example, we call a sequence of functions $F_0, F_1, F_2, \ldots$ a *bi-Hamiltonian* hierarchy if

$$\{F_i, \cdot\}_1 = \{F_{i+1}, \cdot\}_2$$

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If we assume in addition that $\{F_0, \cdot\}_2 = 0$, these conditions together mean that we require $F_\lambda = F_0 + F_1 \lambda + F_2 \lambda^2 + \ldots$ to be in the Poisson centre for the generic bracket $\lambda \{\cdot, \cdot\}_1 - \{\cdot, \cdot\}_2$. 
**Bi-Hamiltonian systems — 2**

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(Lenard ’67, published by Gardner–Greene–Kruskal–Miura ’74, studied in depth by Magri ’78 and Gel’fand–Dorfman ’79. . .)
**Example: KdV**

The Korteweg–de Vries (KdV) equation

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where \( H_0 = \int (u^3 - \frac{1}{2}u_x^2) \, dx \) and \( P_0 = \frac{\partial}{\partial x} \) defines the Poisson bracket

\[ \{F, G\}_0 = \int \frac{\delta F}{\delta u} P_0 \frac{\delta G}{\delta u} \, dx. \]

(\( \frac{\delta}{\delta u} \) is the Fréchet derivative, \( \frac{\delta \int f \, dx}{\delta u} = \sum_{k \geq 0} (-\frac{d}{dx})^k \frac{\partial f}{\partial u^{(k)}} \).)
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Also, the KdV equation can be rewritten as

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Furthermore, Lenard recursion formula

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applied to \( F_0 = \int \frac{u}{2} \, dx, \quad F_1 = H_1 = \int \frac{u^2}{2} \, dx, \) gives the commuting family

Note that \( \delta F_0 \delta u = \frac{1}{2}, \) so \( F_0 \) is in the Poisson centre for \( \{ \cdot, \cdot \}_0. \)
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Examples arising from Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. When is the constant bracket arising from $\gamma \in \Lambda^2 \mathfrak{g}^*$ compatible with the Kostant–Kirillov bracket?

Remark 1: if $\gamma$ is a coboundary, $\gamma(\mathfrak{g}_1, \mathfrak{g}_2) = \zeta(\lbrack \mathfrak{g}_1, \mathfrak{g}_2 \rbrack)$ for some $\zeta \in \mathfrak{g}^*$, then $F_\lambda(\xi) = F(\xi + \lambda \zeta)$ is in the generic centre, if $F$ is in the Poisson centre for the Kostant–Kirillov bracket $\rightrightarrows$ shift of argument method (Mishchenko–Fomenko ’79, Vinberg ’90 . . . ). Recently used to study generalized Gaudin hamiltonians (Rybnikov ’06), $\mathcal{G}$-opers with irregular singularity (Feigin–Frenkel–Rybnikov ’07) etc.

Remark 2: if $\mathfrak{g}$ is the algebra of vector fields on the circle, and $\gamma$ is the 2-cocycle defining the Virasoro algebra, one obtains the above bi-Hamiltonian interpretation of KdV. ($P_0 = \partial / \partial x$ produces the Kostant–Kirillov bracket, while $P_1 = \partial^3 / \partial x^3 + 4u \partial / \partial x + 2ux$ is responsible for the Virasoro cocycle.)
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**Definition:** \( \mathfrak{g} \) is an algebra with two compatible Lie brackets, if there are two brackets \( \{\cdot, \cdot\}_1 \) and \( \{\cdot, \cdot\}_2 \) on \( \mathfrak{g} \) such that \( a\{\cdot, \cdot\}_1 + b\{\cdot, \cdot\}_2 \) is a Lie bracket for any choice of scalars \( a, b \).
**Definition:** $\mathfrak{g}$ is an algebra with two compatible Lie brackets, if there are two brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ on $\mathfrak{g}$ such that $a\{\cdot, \cdot\}_1 + b\{\cdot, \cdot\}_2$ is a Lie bracket for any choice of scalars $a, b$.

If $\mathfrak{g}$ is an algebra with two compatible Lie brackets, $\mathfrak{g}^*$ has a natural structure of a bi-Hamiltonian manifold: two Kostant–Kirillov brackets are compatible.
In the case of one Lie bracket, one can obtain Lie algebras taking commutators of associative algebras: if $A$ is an associative algebra, then the operation $ab - ba$ defines a Lie bracket on $A$. 

Claim: if $A$ is an algebra with two compatible products, then the operations $(a \star_1 b) - (b \star_1 a)$ and $(a \star_2 b) - (b \star_2 a)$ make it an algebra with two compatible Lie brackets.
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Similarly, we call $A$ an algebra with two compatible products, if there are two products $(\cdot \ast_1 \cdot)$ and $(\cdot \ast_2 \cdot)$ on $A$ such that the product $a(\cdot \ast_1 \cdot) + b(\cdot \ast_2 \cdot)$ is associative for any choice of scalars $a, b$. 
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Exercise: check the compatibility (and find a conceptual explanation).
Further Examples

Example: $A_k$ is the space of polynomials in $t$ of degree at most $k - 1$, $\alpha_1(t)$ and $\alpha_2(t)$ two polynomials of degree $k$ without common roots. Then, by Euclid,

$$f(t)g(t) = x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)$$

for unique pair $(x_1(t), x_2(t)) \in A_k \times A_k$. We let $f(t) \ast_i g(t) = x_i(t)$. 

This has important generalizations for elliptic curves where polynomials are replaced by matrix-valued $\theta$-functions. (Odesskii–Sokolov '05)

Example: If $A$ is an algebra with two compatible products and $B$ is an associative algebra then $A \otimes B$ is an algebra with two compatible products. Consequently, $\text{Mat}_n(A)$ is an algebra with two compatible brackets. This descends on $\text{GL}_n$-invariants, and leads to bi-Hamiltonian interpretations of some "integrable ODEs on associative algebras" (Olver–Sokolov '98, Mikhailov–Sokolov '99).
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Further examples

Example: $A_k$ is the space of polynomials in $t$ of degree at most $k - 1$, $\alpha_1(t)$ and $\alpha_2(t)$ two polynomials of degree $k$ without common roots. Then, by Euclid,

$$f(t)g(t) = x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)$$

for unique pair $(x_1(t), x_2(t)) \in A_k \times A_k$. We let $f(t) \star_i g(t) = x_i(t)$. This has important generalizations for elliptic curves where polynomials are replaced by matrix-valued $\theta$-functions. (Odesskii–Sokolov ’05)

Example: If $A$ is an algebra with two compatible products and $B$ is an associative algebra then $A \otimes B$ is an algebra with two compatible products. Consequently, $\text{Mat}_n(A)$ is an algebra with two compatible brackets. This descends on $GL_n$-invariants, and leads to bi-Hamiltonian interpretations of some “integrable ODEs on associative algebras” (Olver–Sokolov ’98, Mikhailov–Sokolov ’99).
Results on algebras with two compatible products

Towards the classification: Odesskii–Sokolov ’05, classification of ways to make $\bigoplus_i \text{Mat}_{n_i}$ an algebra with two compatible products. The arising combinatorial data is a representation of a quiver associated to an affine simply laced Dynkin diagram (with some additional marking of vertices).
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**On the other extremal side:** free algebras with two compatible products. For one product, the free algebra is just a tensor algebra. Is there a simple description for the case of two products?

YES!
Denote by $RT(S)$ the collection of all planar rooted trees whose non-root vertices are labelled by elements of a finite set $S$. 
Main theorem

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**Theorem (D. ’08)**

The vector space $\k \RT(S)$ has two compatible products; with those products it becomes the free algebra with two compatible products generated by $S$.
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Theorem (D. ’08)

The vector space $k \text{RT}(S)$ has two compatible products; with those products it becomes the free algebra with two compatible products generated by $S$.

Corollary (on Catalan numbers)

The component of degree $n$ in the free algebra with two compatible brackets generated by $S$ has dimension $\frac{1}{n+1} \binom{2n}{n} |S|^n$. 
Let $T_1, T_2 \in RT(S)$. Assume that the root of $T_1$ has $k$ children. Define the products $T_1 \star_1 T_2$ and $T_1 \star_2 T_2$ as follows:
Compatible products of trees

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$$T_1 \star_1 T_2 = \sum_{f : [k] \to \text{Vertices}(T_2)} T_1 \triangleright^f T_2,$$

where $T_1 \triangleright^f T_2$ is obtained as follows. We split $T_1$ into $k$ parts $T_1[1], \ldots, T_1[k]$, and for a vertex $v$ of $T_2$ with $f^{-1}(v) = \{i_1, \ldots, i_s\}$, we graft $T_1[i_1], \ldots, T_1[i_s]$ at vertex $v$ (keeping the label of $v$) to the left of all the children of $v$ in $T_2$. 
Compatible products of trees

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we graft $T_1[i_1], \ldots, T_1[i_s]$ at vertex $v$ (keeping the label of $v$) to the left of all the children of $v$ in $T_2$. 
**Example**

For the trees

\[ T_1 = a, \quad T_2 = b \lor c \]

the product \( T_1 \star T_2 \) is equal to

\[ a \lor b \lor c + a \lor b \lor c \]

while the product \( T_1 \star T_2 \) is equal to

\[ a \lor b \lor c \]
Example

For the trees

\[ T_1 = b, \quad T_2 = a \]

the product \( T_1 \star_1 T_2 \) is equal to

\[ + b + a \]

\[ a \]
**Example**

For the trees

\[ T_1 = \begin{array}{c} a \\ \end{array}, \quad T_2 = \begin{array}{c} b \\ \end{array} \]

the product \( T_1 \star_1 T_2 \) is equal to

\[ \begin{array}{ccc} a & b & c \\ a \\ \end{array} + \begin{array}{ccc} b & c \\ b \\ \end{array} + \begin{array}{ccc} c \\ a \end{array} \]

while the product \( T_1 \star_2 T_2 \) is equal to

\[ \begin{array}{c} a \\ \end{array} + \begin{array}{c} b \\ \end{array} + \begin{array}{c} c \\ \end{array} \]
Observation 1: The compatibility condition can be rewritten in the form

$$(T_1 \ast_2 T_2) \ast_1 T_3 - T_1 \ast_2 (T_2 \ast_1 T_3) = T_1 \ast_1 (T_2 \ast_2 T_3) - (T_1 \ast_1 T_2) \ast_2 T_3.$$
Proof strategy for compatibility

**Observation 1:** The compatibility condition can be rewritten in the form

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**Observation 2:** For two products on the space of trees that we defined both the left hand side and the right hand side have positive integer coefficients.
**Example**

For the trees

\[ T_1 = \quad , \quad T_2 = \quad , \quad T_3 = \quad \]

\[
T_1 = a , \quad T_2 = b \quad c , \quad T_3 = d
\]

The product \((T_1 \star T_2 \star T_3)\) is

\[
a \quad b \quad c \quad d
+ a \quad b \quad c
d
+ a \quad b \quad c
d
+ b \quad a \quad c
d
+ c \quad a \quad b
d
+ c \quad a \quad b
d
+ b \quad a \quad c
d
+ a \quad b \quad c
d
\]
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For the trees

\[ T_1 = \quad , \quad T_2 = \quad , \quad T_3 = \quad \]

the product \( (T_1 \star_2 T_2) \star_1 T_3 \) is

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]
**Example**

For the trees

\[ T_1 = \begin{array}{c} \cdot \\ a \end{array}, \quad T_2 = \begin{array}{c} \cdot \\ b \\ \cdot \\ c \\ \cdot \\ d \end{array}, \quad T_3 = \begin{array}{c} \cdot \\ a \end{array} \]

the product \( (T_1 \star_2 T_2) \star_1 T_3 \) is

\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
\end{array}
\]

the product \( T_1 \star_2 (T_2 \star_1 T_3) \) is

\[
\begin{array}{c}
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
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c \\
d
\end{array}
+ \\
\begin{array}{c}
\cdot \\
\cdot \\
a \\
b \\
\cdot \\
c \\
d
\end{array}
+ \\
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\cdot \\
\cdot \\
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b \\
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c \\
d
\end{array}
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\]
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Also, the product $T_1 \star_1 (T_2 \star_2 T_3)$ is
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Proof of compatibility

In the compatibility condition

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the trees that appear on the left hand side are those for which there exist subtrees of \(T_1\) that are attached to some leaves of \(T_3\).
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the trees that appear on the left hand side are those for which there exist subtrees of \(T_1\) that are attached to some leaves of \(T_3\). The right hand side has the same interpretation.
Proof of freeness

The proof of freeness goes in two steps:
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Proof by induction, rather easy.

Free algebra with two compatible products generated by $S$ has the same dimensions of graded components as $\mathbb{k}RT(S)$.

Original proof was using a result of Strohmayer '07 that involved heavy homological machinery (Koszul duality for operads), however now it is clear that using Gr"obner bases for operads (D.–Khoroshkin '08) can simplify the proof substantially.
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The first product $T_1 \star_1 T_2$ on $\mathbb{Q}RT(S)$ was defined by Grossman and Larson ’89 in their works about Hopf algebras based on trees (combinatorial Hopf algebras describing solving differential equations etc.). Our results yield an easy proof of their theorem stating that the algebra of planar rooted trees is a free associative algebra generated by trees whose root has only one child.
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Our strategy to construct a compatible product (grafting to internal vertices only) applies to a variety of cases of algebras based on trees, e.g. numerous “Hopf algebras of renormalization” (Brouder–Frabetti ’03, Connes–Kreimer ’98, Foissy ’02).
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(Vague) question: relate renormalization techniques to bi-Hamiltonian integrability.
Free bi-Hamiltonian algebras

What about free algebras with two compatible Lie brackets (a more natural candidate to study)?
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**Bad news:** similarly to the case of one product/bracket, where free Lie algebras are more complicated than free associative algebras, for free algebras with compatible Lie brackets both the dimension formulas and combinatorics are quite disastrous.

**Good news:** the operadic part (the space of multilinear elements in the free algebra with \( n \) generators) is a very interesting object. It has dimension \( n^{n-1} \) and interesting combinatorics (D.–Khoroshkin ’06, Liu ’09).

**Even better news:** for free bi-Hamiltonian algebras, the corresponding dimensions are \((n+1)^{n-1}\) (op. cit.), which makes one think of diagonal harmonics. . .
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Thank you for your patience!