ANICK-TYPE RESOLUTIONS, SHUFFLE ALGEBRAS, AND CONSECUTIVE PATTERN AVOIDANCE

Vladimir Dotsenko

Dublin Institute for Advanced Studies and Trinity College Dublin

joint work with Anton Khoroshkin (ETH Zurich)

arXiv:1002.2761

British Mathematics Colloquium, Edinburgh

April 6, 2010
WORD AVOIDANCE IN REAL LIFE
Word avoidance in real life
Problem:
Enumerate words of length $N$ which do not contain a subword SEX.
Avoidance of SEX

Problem:
Enumerate words of length $N$ which do not contain a subword SEX.

Solution:

SEX-less words = (all words) −
− (words with at least one subword SEX) +
+ (words with at least two subwords SEX) − ...
Avoidance of *SEX*

**Problem:**
Enumerate words of length \( N \) which do not contain a subword *SEX*.

**Solution:**

\[
\text{SEX-less words} = \text{(all words)} - \left( \text{words with at least one subword } \text{SEX} \right) + \left( \text{words with at least two subwords } \text{SEX} \right) - \ldots
\]

which easily yields a formula for generating functions

\[
f_{\text{no-SEX}}(t) = \frac{1}{1 - (26t + y)} \bigg|_{y = -t^3} = \frac{1}{1 - 26t + t^3}.
\]
Avoidance of Sex and Expert

Problem:
Enumerate words of length \( N \) which do not contain either a subword Sex or a subword Expert.
Avoidance of SEX and EXPERT

Problem:
Enumerate words of length $N$ which do not contain either a subword SEX or a subword EXPERT.

Solution: Same inclusion–exclusion argument gives the formula

$$f_{\text{no-SEX, no-EXPERT}}(t) = \frac{1}{1 - 26t + t^3 + t^6 - t^7},$$
Avoidance of **SEX** and **EXPERT**

**Problem:**
Enumerate words of length $N$ which do not contain either a subword **SEX** or a subword **EXPERT**.

**Solution:** Same inclusion–exclusion argument gives the formula

$$f_{\text{no-SEX, no-EXPERT}}(t) = \frac{1}{1 - 26t + t^3 + t^6 - t^7},$$

where

$$t^3 = \text{weight}(\text{SEX}), t^6 = \text{weight}(\text{EXPERT}), t^7 = \text{weight}(\text{SEXPERT}).$$
Avoidance of SEX and EXPERT

Problem:
Enumerate words of length \( N \) which do not contain either a subword SEX or a subword EXPERT.

Solution: Same inclusion–exclusion argument gives the formula

\[
f_{\text{no–SEX, no–EXPERT}}(t) = \frac{1}{1 - 26t + t^3 + t^6 - t^7},
\]

where

\[ t^3 = \text{weight(SEX)}, \ t^6 = \text{weight(EXPERT)}, \ t^7 = \text{weight(SEXPERT)}. \]

Here

\[ \text{SEXPERT} = \left\{ \begin{array}{c} \text{SEX} \\ \text{EXPERT} \end{array} \right\} \]

is a cluster.
**Theorem (I. P. Goulden & D. M. Jackson ’79):**
Let $P$ be a set of illegal words in the alphabet $X$. Then

$$f_{\text{no-}P}(t) = \frac{1}{1 - |X|t + Cl_P(t, -1)},$$

where $Cl_P(t, s) = \sum \text{cl}^P_{n, m} t^n s^m$ counts clusters ($\text{cl}^P_{n, m}$ is the number of clusters on $n$ letters formed by $m$ words from $P$).
Avoidance of SEX and EXPERTISE

Problem:
Enumerate words of length $N$ which do not contain either a subword SEX or a subword EXPERTISE.
**Avoidance of SEX and EXPERTISE**

**Problem:**
Enumerate words of length $N$ which do not contain either a subword SEX or a subword EXPERTISE.

Here we have infinitely many clusters, e.g. EXPERTISE, EXPERTISEEXPERTISE, EXPERTISEEXPERTISEEXPERTISE etc.
Avoidance of SEX and EXPERTISE

Problem:
Enumerate words of length $N$ which do not contain either a subword SEX or a subword EXPERTISE.

Here we have infinitely many clusters, e.g. EXPERTISE, EXPERTISEEXPERTISE, EXPERTISEEXPERTISEEXPERTISE etc.

Moreover, some words admit many different coverings, e.g. we have the following two clusters

\[
\left\{ \text{EXPERTISE} \right\}, \quad \left\{ \text{EXPERTISE} \right\}.
\]
**Avoidance of SEX and EXPERTISE**

**Observation:** Contributions of the two clusters

\[
\begin{align*}
\{ & \text{EXPERTISE} \\
& \text{EXPERTISE} \} \quad \text{and} \quad \begin{Bmatrix}
\text{EXPERTISE} \\
\text{SEX} \\
\text{EXPERTISE} \end{Bmatrix}
\end{align*}
\]

cancel each other because the first one is formed by two illegal words, and the second one — by three.
Avoidance of SEX and EXPERTISE

**Observation:** Contributions of the two clusters

\[
\left\{ \text{EXPERTISE} \right\} \quad \text{and} \quad \left\{ \text{SEX} \right\}.
\]

cancel each other because the first one is formed by two illegal words, and the second one — by three.

**After cancellations:** clusters that contribute are SEX, EXPERTISE, SEXPERTISE, EXPERTISEX, SEXPERTISEX, so that

\[
f_{\text{no-SEX, no-EXPERTISE}}(t) = \frac{1}{1 - 26t + t^3 + t^9 - 2t^{10} + t^{11}}.
\]
Anick chains

**Question:** how to describe clusters that survive after those obvious cancellations?
Question: how to describe clusters that survive after those obvious cancellations?

Answer (D. J. Anick ’86): chains, defined as follows:
**Question:** how to describe clusters that survive after those obvious cancellations?

**Answer (D. J. Anick ’86):** chains, defined as follows:
— a single letter is a 0-chain;

Example: EXPERTISEXPERTISE, even though can be represented as a link of two illegal words, is not a 2-chain because its proper beginning EXPERTISEX is already a 2-chain! It’s not a 3-chain either, because the first and the third illegal words are linked.
**Anick chains**

**Question:** how to describe clusters that survive after those obvious cancellations?

**Answer (D. J. Anick ’86):** chains, defined as follows:
— a single letter is a 0-chain;
— an $m$-chain is obtained by linking together $m$ illegal words so that only neighbours are linked, the first $(m − 1)$ illegal words form an $(m − 1)$-chain, and no proper beginning forms an $m$-chain.

Example: EXPERTISEXPERTISE, even though can be represented as a link of two illegal words, is not a 2-chain because its proper beginning EXPERTISEX is already a 2-chain! It's not a 3-chain either, because the first and the third illegal words are linked.
**Anick chains**

**Question:** how to describe clusters that survive after those obvious cancellations?

**Answer (D. J. Anick ’86):** chains, defined as follows:
— a single letter is a 0-chain;
— an $m$-chain is obtained by linking together $m$ illegal words so that only neighbours are linked, the first $(m-1)$ illegal words form an $(m-1)$-chain, and no proper beginning forms an $m$-chain.

**Example:**
EXPERTISEEXPERTISE, even though can be represented as a link of two illegal words, is not a 2-chain because its proper beginning EXPERTISEX is already a 2-chain! It’s not a 3-chain either, because the first and the third illegal words are linked.
Theorem (D. J. Anick ’86): We have

\[ f_{\text{no-P}}(t) = \frac{1}{1 - |X|t + C_P(t, -1)}, \]

where \( C_P(t, s) = \sum c_{n,m}^P t^n s^m \) counts chains (\( c_{n,m}^P \) is the number of \( m \)-chains on \( n \) letters).
Anick resolution

**Proof:** Denote by $A$ the associative algebra with generators $X$ and relations $P = 0$. Also, denote by $C_m$ the vector space with a basis of $m$-chains. Then there exists a chain complex

\[ \ldots \to C_n \otimes A \to C_{n-1} \otimes A \to \ldots \to C_1 \otimes A \to C_0 \otimes A \to A \to 0, \]

whose homology is concentrated in the rightmost term and is one-dimensional. Boundary maps move “tails” through the tensor product: $\partial(w't \otimes a) = w' \otimes ta$. 

Compute (graded) Euler characteristics of this complex:

\[ (1 - C_0(t) + C_1(t) - \ldots) A(t) = 1. \]

Clearly, $1 - C_0(t) + C_1(t) - \ldots = 1 - mt + C_P(t) - 1$, and $A(t)$ enumerates words that avoid $P$. 
Proof: Denote by $A$ the associative algebra with generators $X$ and relations $P = 0$. Also, denote by $C_m$ the vector space with a basis of $m$-chains. Then there exists a chain complex

$$\ldots \rightarrow C_n \otimes A \rightarrow C_{n-1} \otimes A \rightarrow \ldots \rightarrow C_1 \otimes A \rightarrow C_0 \otimes A \rightarrow A \rightarrow 0,$$

whose homology is concentrated in the rightmost term and is one-dimensional. Boundary maps move “tails” through the tensor product: $\partial(w't \otimes a) = w' \otimes ta$.

Compute (graded) Euler characteristics of this complex:

$$(1 - C_0(t) + C_1(t) - \ldots)A(t) = 1.$$ 

Clearly, $1 - C_0(t) + C_1(t) - \ldots = 1 - mt + C_P(t, -1)$, and $A(t)$ enumerates words that avoid $P$. □
Definition: Let $\sigma \in S_n$, $\tau \in S_m$ be permutations. We say that $\sigma$ contains $\tau$ as a consecutive pattern if a subword of $\sigma$ is order-isomorphic to $\tau$. Otherwise we say that $\sigma$ avoids $\tau$. For example, $132$ is contained in $41532$ (since $153$ is order-isomorphic to $132$), and is avoided by $52134$. For enumeration, exponential generating functions are used, e.g. $f_{\text{no}-132}(t) = 1 + \sum_{n \geq 1} a_{\text{no}-132}(n) \frac{n!}{n!} t^n$. 
Definition: Let $\sigma \in S_n$, $\tau \in S_m$ be permutations. We say that $\sigma$ contains $\tau$ as a consecutive pattern if a subword of $\sigma$ is order-isomorphic to $\tau$. Otherwise we say that $\sigma$ avoids $\tau$.

For example, 132 is contained in 41532 (since 153 is order-isomorphic to 132), and is avoided by 52134.
Definition: Let $\sigma \in S_n$, $\tau \in S_m$ be permutations. We say that $\sigma$ contains $\tau$ as a consecutive pattern if a subword of $\sigma$ is order-isomorphic to $\tau$. Otherwise we say that $\sigma$ avoids $\tau$.

For example, 132 is contained in 41532 (since 153 is order-isomorphic to 132), and is avoided by 52134.

For enumeration, exponential generating functions are used, e.g.

$$f_{\text{no}-132}(t) = 1 + \sum_{n \geq 1} \frac{a_{\text{no}-132}(n)}{n!} t^n.$$
Theorem (I. P. Goulden & D. M. Jackson ’79):

\[ f_{\text{no-123}}(t) = \frac{1}{1 - t + \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \ldots}. \]
Pattern avoidance in permutations

Theorem (I. P. Goulden & D. M. Jackson ’79):

\[ f_{\text{no}-123}(t) = \frac{1}{1 - t + \frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \ldots}. \]

Theorem (S. Elizalde & M. Noy ’03):

\[ f_{\text{no}-132}(t) = \frac{1}{1 - \int_0^t e^{-u^2/2} \, du}. \]
Shuffle product of graded vector spaces

**Wanted:** a materialization on the level of vector spaces for the product of *exponential* generating functions; on the level of coefficients,

\[ c_n = \sum_k \binom{n}{k} a_k b_{n-k}. \]
**Shuffle product of graded vector spaces**

**Wanted:** a materialization on the level of vector spaces for the product of *exponential* generating functions; on the level of coefficients,

\[ c_n = \sum_k \binom{n}{k} a_k b_{n-k}. \]

**Claim:** Such a product of vector spaces exists!
**Shuffle product of graded vector spaces**

**Wanted:** a materialization on the level of vector spaces for the product of exponential generating functions; on the level of coefficients,

\[ c_n = \sum_k \binom{n}{k} a_k b_{n-k}. \]

**Claim:** Such a product of vector spaces exists!

For two graded \( k \)-vector spaces \( A = \bigoplus_{n \geq 1} A_n \) and \( B = \bigoplus_{n \geq 1} B_n \), their shuffle product \( A \boxtimes B \) is defined as the graded vector space \( C = \bigoplus_{n \geq 1} C_n \) with

\[ C_n = \bigoplus_{k+l=n} k\text{Sh}(k, l) \otimes A_k \otimes B_l, \]

where \( \text{Sh}(k, l) \) is the set of all \((k, l)\)-shuffles in \( S_n \). It’s what we want for generating functions, since \( |\text{Sh}(k, l)| = \binom{k+l}{k} \).
Shuffle algebras

Definition (M. Ronco ’07): A shuffle algebra is a graded vector space with an associative product $A \boxtimes A \rightarrow A$.

Example: The vector space $\bigoplus n k S_n$ is a free shuffle algebra with one generator.

Generalisation: let $P$ be a set of illegal patterns, and let $A_n, P$ be the linear span in $k S_n$ of all $P$-avoiding permutations. Then $A_P$ is a shuffle algebra which is the quotient of the free algebra by the ideal generated by $P$.

If we start with the free shuffle algebra with several generators, we shall end up with the notion of coloured patterns (Mansour ’01); all our further statements remain.
Shuffle algebras

Definition (M. Ronco ’07): A shuffle algebra is a graded vector space with an associative product $A \boxtimes A \rightarrow A$.

Example: The vector space

$$\bigoplus_{n} \mathbb{k}S_n$$

is a free shuffle algebra with one generator.
**Definition (M. Ronco ’07):** A shuffle algebra is a graded vector space with an associative product $A \boxtimes A \rightarrow A$.

**Example:** The vector space

$$\bigoplus_n \mathbb{k}S_n$$

is a *free* shuffle algebra with one generator.

**Generalisation:** let $P$ be a set of illegal patterns, and let $A_{n,P}$ be the linear span in $\mathbb{k}S_n$ of all $P$-avoiding permutations. Then $A_P$ is a shuffle algebra which is the quotient of the free algebra by the ideal generated by $P$. 
Shuffle algebras

**Definition (M. Ronco ’07):** A shuffle algebra is a graded vector space with an associative product $A \boxtimes A \rightarrow A$.

**Example:** The vector space

$$\bigoplus_n \mathbb{k}S_n$$

is a *free* shuffle algebra with one generator.

**Generalisation:** let $P$ be a set of illegal patterns, and let $A_{n,P}$ be the linear span in $\mathbb{k}S_n$ of all $P$-avoiding permutations. Then $A_P$ is a shuffle algebra which is the quotient of the free algebra by the ideal generated by $P$.

If we start with the free shuffle algebra with several generators, we shall end up with the notion of *coloured patterns* (Mansour ’01); all our further statements remain.
Chains in the context of permutations are defined as follows:
Anick-type chains

Chains in the context of permutations are defined as follows:
— the only permutation of one element is a 0-chain;
Chains in the context of permutations are defined as follows:
— the only permutation of one element is a 0-chain;
— an $m$-chain is obtained by linking together $m$ illegal patterns so that only neighbours are linked, the first $(m - 1)$ illegal patterns form an $(m - 1)$-chain (up to order-iso), and no proper beginning forms an $m$-chain.
Anick-type chains

Chains in the context of permutations are defined as follows:
— the only permutation of one element is a 0-chain;
— an $m$-chain is obtained by linking together $m$ illegal patterns so that only neighbours are linked, the first $(m - 1)$ illegal patterns form an $(m - 1)$-chain (up to order-iso), and no proper beginning forms an $m$-chain.

Example: for $P = \{123\}$ we get 1, 123, \( \begin{cases} 123 \\ 234 \end{cases} \), \( \begin{cases} 123 \\ 234 \\ 456 \end{cases} \), ... 

Note that 12345 is neither a 2-chain (as 1234 is already a 2-chain) nor a 3-chain (as 123 and 345 are linked).
Denote by $A$ the shuffle algebra with one generator whose relations are all illegal patterns. Also, denote by $C_m$ the vector space with a basis of $m$-chains. Then there exists a chain complex

$$\ldots \rightarrow C_n \boxtimes A \rightarrow C_{n-1} \boxtimes A \rightarrow \ldots \rightarrow C_1 \boxtimes A \rightarrow C_0 \boxtimes A \rightarrow A \rightarrow 0,$$

whose homology is concentrated in the rightmost term and is one-dimensional. Boundary maps move “tails” through the shuffle product.
Consequently, we proved the following

\begin{equation}
\sum_{c P(n, m)} t^n s^m = \frac{1}{1 - t} + C P(t, s),
\end{equation}

where \( C P(t, s) \) is the exponential generating function counting chains (\( c P(n, m) \) is the number of \( m \)-chains on \( n \) letters).

Many corollaries, for example, a proof of the following Conjecture (S. Elizalde '03):

For a pattern \( \tau \) without self-overlaps, the number of permutations avoiding \( \tau \) depends only on the first and the last element of \( \tau \).
Consequently, we proved the following

**Theorem:** We have

\[ f_{\text{no-P}}(t) = \frac{1}{1 - t + C_P(t, -1)}, \]

where

\[ C_P(t, s) = \sum c_{n,m}^P \frac{t^n}{n!} s^m \]

is the exponential generating function counting chains (\( c_{n,m}^P \) is the number of \( m \)-chains on \( n \) letters).
Consequently, we proved the following

**Theorem:** We have

\[ f_{\text{no-P}}(t) = \frac{1}{1 - t + C_P(t, -1)}, \]

where

\[ C_P(t, s) = \sum c_{n,m}^{P} \frac{t^n}{n!} s^m \]

is the exponential generating function counting chains (\(c_{n,m}^{P}\) is the number of \(m\)-chains on \(n\) letters).

Many corollaries, for example, a proof of the following
Consequently, we proved the following

**Theorem:** We have

\[ f_{\text{no-P}}(t) = \frac{1}{1 - t + C_P(t, -1)}, \]

where

\[ C_P(t, s) = \sum c_{n,m}^P \frac{t^n}{n!} s^m \]

is the exponential generating function counting chains (\( c_{n,m}^P \) is the number of \( m \)-chains on \( n \) letters).

Many corollaries, for example, a proof of the following

**Conjecture (S. Elizalde '03):** For a pattern \( \tau \) without self-overlaps, the number of permutations avoiding \( \tau \) depends only on the first and the last element of \( \tau \).
Thank you for your patience!