Associahedra to ∞ and beyond

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Geometry and combinatorics of associativity
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The *so-called* Stasheff polytope was in fact constructed by Tamari in 1951, a full decade before my version.

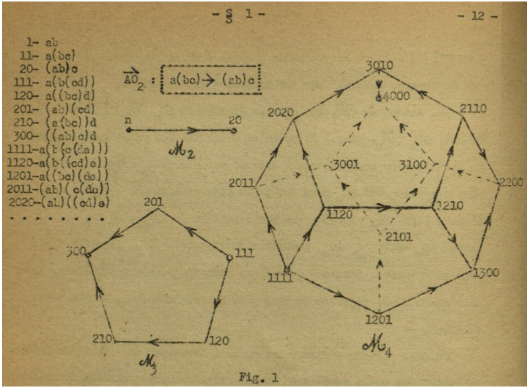
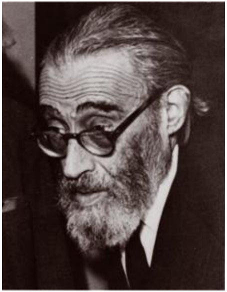


Fig. 1

Beginnings

For me, associativity began as an undergraduate at Michigan. I was privileged to have a course in classical projective geometry from George Yuri Rainich[1].



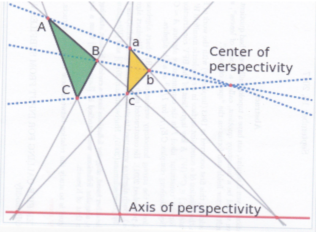
As I've said elsewhere, he 'first revealed to me the deep significance yet optional nature of associativity.'

[1] born Yuri Germanovich Rabinovich

History

It was Desargues's theorem in projective geometry that Rainich drew to my attention:

Corresponding sides of two triangles, when extended, meet at points on a line called the axis of perspectivity. The lines which run through corresponding vertices on the triangles meet at a point called the center of perspectivity.



Desargues's theorem states that the truth of the first condition is necessary and sufficient for the truth of the second.

History

In projective 3-space, Desargues's theorem is always true.

For a projective plane in which Desargues's theorem is true, an underlying algebraic coordinate system must be a division ring (skewfield); in particular associative.

The relation between projective 3-space and associativity is more subtle in the setting of H-spaces.

History

As a grad student at Princeton (1956-57), I was fortunate to be the scribe for Milnor's course on Characteristic Classes, but when he went on leave (1957-58), I was again fortunate to be passed to John Moore who led me into the study of H-spaces.

Particularly important was my reading of work by Masahiro Sugawara , especially *A condition that a space is group-like*.

I had already encountered homotopy associativity in Hilton's " An introduction to Homotopy Theory",
but Sugawara showed:

Theorem

A CW H-space (X, m) *is admits a homotopy for associativity iff the Hopf fibration* $X \to X * X \to SX$ *admits an extension* $X \to E_2 \to XP(2)$ *with* $X * X$ *contractible in* E_2*.*

History

Recall that a *projective plane* $XP(2)$ for an H-space (X, m) is constructed as the mapping cone $SX \cup_m C(X * X)$ of the Hopf fibration.

Key to Sugawar's construction is the action of X on $X * X$ via an *associating homotopy* $h_t : X^3 \to X$ from $(xy)z$ at 0 to $x(yz)$ at 1 using a specific formula:

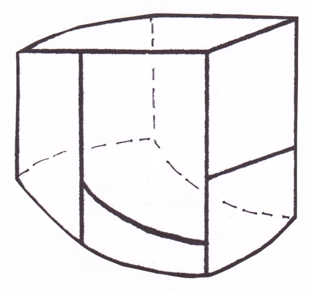
$$x(y,t,z) = \begin{cases} (xy, 2t, z) & \text{for } 0 \leq t \leq 1/2 \\ h_{2t-1}(x,y,z) & \text{for } 01/2 \leq t \leq 1. \end{cases} \tag{1}$$

This explains why the ubiquitous pentagon appeared for me as

rather than the symmetric regular pentagon of Tamari, Mac Lane

History

Similarly the associahedron K_5 appears as



rather than as in Tamari's thesis.

History

The sssociahedra K_n were introduced to define A_n-spaces with H-spaces being A_2-spaces, leading to the basic:

Theorem

A connected CW X admits the structure of an A_∞-space iff X has the homotopy type of a based loop space ΩY.

There was a notion of equivalence up to strong homotopy (in an appropriate sense) for A_∞-structures on X so that equivalence classes corresponded to homotopy types of such spaces Y.

By analogy with the classifying space BG of a topological group, $XP(\infty)$ is often denoted BX.

Thus was the ∞-langauge born, the alternative being *strong homotopy* or *sh*; both are useful.

History

For H-spaces X_1, X_2, the appropriate notion of morphism is that of homotopy multiplicative map, but for A_∞-spaces?

Again Sugawara (1960/61) led the way, introducing:

Definition

For strictly associative H-space X_1, X_2, a map $M_1 : X_1 \to X_2$ is strongly homotopy multiplicative if there exist maps $M_n : (X_1)^n \times I^n \to X_2$ such that

$$M_n(x_1, \cdots, x_n, t_1, \cdots t_n) =$$
$$= M_{n-1}(\cdots, x_{i-1}x_i, \cdots) \text{ for } t_i = 0$$
$$= M_{i-1}(\cdots, x_{i-1}, \cdots, t_{i-1})M_{n-i}(x_i, \cdots, x_n, t_{i+1}, \cdots, t_n) \text{ for } t_i = 1.$$

For A_∞-spaces, life is much more complicated

'Yoga'

These examples illustrate two different 'philosophies'
(or as Bott would say: 'yogas'):

The *geometric combinatorial* is most visible in the various realizations of the associahedra as convex polytopes in the strict sense. Perhaps even as a hydrocarbon "associahedrane" $C_{14}H_{14}$!! (similar to the Platonic hydrocarbons).

Google hits for associahedra were about 45,400, together with numerous graphic versions, including animations. There are also models in various 3D media.

Stefan has assembled a marvelous *Hedra Zoo*.
(www.math.uakron.edu/ sf34/hedra.htm)

The *algebraic/topological* was developed further first.

Algebraic/Topological

An obvious next step was to invoke commutativity, knowing 2-fold loop spaces $\Omega^2 Z$ were homotopy commutative.

Again Sugawara' s strongly homotopy multiplicative maps provided a lead. Just assuming homotopy commutativity, I found

Theorem

If an H-space (X, m) is homotopy commutative, there is a map $SX \times SX \to XP(2)$ which is the inclusion on each factor.

Trying to extend this bare-handed approach to an H-space structure on BX was too laborious to carry out, until Peter May provided a way forward with his concept of *operad*:

Algebraic/Topological

Definition

An operad $(\mathcal{O}, \circ_i)$ *consists of a collection of* Σ_n*-objects* $\mathcal{O}(n)$ *and maps* $\circ_i : \mathcal{O}(m) \times \mathcal{O}(n) \to \mathcal{O}(m+n-1)$ *for* $m, n \geq 1$ *satisfying the relations visible in the example* $End_X = Map(X^n, X)$.

In most examples, the structures are 'manifest' without appeal to the technical definitions; as Frank Adams used to say, to operate the machine, it is not necessary to raise the bonnet (look under the hood).

In particular, the associahedra form
an operad by freely adding the Σ_n actions, using $\Sigma_n \times K_n$. Alternatively, one refers to non-symmetric operads.

Homotopy characterization of iterated loop spaces $\Omega^n X_n$ for some space X_n required the full power of the theory of operads with the symmetries.

Of particular importance is

Definition

The little n-cubes operad $C_n = \{C_n(j)\}_{j \geq 1}$ has each $C_n(j)$ consisting of ordered configurations of j n-cubes linearly embedded in the standard n-dimensional unit cube I^n with disjoint interiors and axes parallel to those of I^n.

Algebraic/Topological

Theorem

May (1972) A connected CW space Y of the homotopy type of an n-fold loop space $\Omega^n X_n$
admits a coherent system of maps $C_n(j) \times Y^j \to Y$.

Later it was realized that the assocciahedra could be parameterized by the centers of the little cubes (intervals) in $C_1(j)$:

1. K_3 by a single point in $[0, 1]$ including 0 and 1,
2. K_4 by a pair of points in $[0, 1]$, the boundary of the pentagon corresponding to collisions of the points, including with 0 or with 1,
3. K_5 by three points in $[0, 1]$.

Kontsevich has a marvelous image of a magnifying class applied to see how the collisions are approached.

Alternatively, the assocciahedron K_n can be realized as the compactification of the space of n distinct points in $[0, 1]$.

L_∞ and physics

The analog of an A_∞-space is an A_∞-algebra; there is a Lie analog, an L_∞-algebra. These two special cases of *higher homotopy algebras* are particularly important in mathematical physics:

A_∞ for open string field theory and L_∞ for closed string field theory, for deformation quantization and various *'field theories'.*

L_∞- algebras appeared independently in the mid-1980's: in a supporting role in deformation theory in my work with Mike Schlessinger on rational homotopy theory and in correspondence between Drinfel'd and Schechtman as well as in mathematical physics.

L_∞- algebras were in fact already implicit in Sullivan's rational homotopy models.

L_∞ and physics

As best I can recall, I came to focus on L_∞-algebras in my work with Mike where I learned the yoga: deformation problems are controlled by dg Lie algebras, more fully, by L_∞-algebras.

The latter and their morphisms were crucial to Kontsevich's celebrated proof of the *Formality conjecture*.

L_∞ and physics

The Lie algebra L that Mike and I constructed for classification of rational homotopy types was huge, so it was helpful to have at hand the L_∞-structure induced on $H(L)$ for any dg Lie algebra L.

Though $H(L)$ inherits a strict dgLie structure (with $d = 0$), over e.g. a field, there is a (highly non-trivial in general) L_∞-structure such that L and $H(L)$ are equivalent as L_∞ algebras.

This transfer of structure began in the case of dg associative algebras with the work of Kadeishvili.

The jump to the L_∞ analog was immediate, once there was a use for it.

L_∞ and physics

With this preparation, I was able to recognize the mathematical structure of L_∞-algebras (aka sh-Lie algebras) appearing in physics as symmetries or gauge transformations.

This happened first in hearing Zwiebach's talk on *closed string field theory* at the third GUT Workshop (at UNC) in 1982.

L_∞ and physics

Then in 1985 in my conversation at UNC with Gerrit Burgers (visiting Henk van Dam, physics UNC-CH), I found we had common formulas, if not a common language.

In their study of *field dependent* gauge symmetries for field theories for higher spin particles, Behrends, Burgers and van Dam invented/discovered what later turned out to be an L_∞-structure. This we worked out by 1997 in Barnich et al. (1998).

L_∞ and physics

Meanwhile in 1987, Gregg Zuckerman told me to look at a paper by Browning and McMullan (1987). There was a diagram which depicted a multi-complex which illustrated the paper(s) of Batalin-Fradkin-Fradkina-Vilkovisky (BFV) from 1975-1985 concerning reduction of constrained Hamiltonian systems.

For a Hamiltonian system with constraints forming a Lie algebra and generating a commutative ideal, their construction combined a Koszul-Tate resolution with a Chevalley-Eilenberg complex. The sum of the differentials no longer squared to 0 but required terms of higher order.

I was able to understand those terms in the context of homological perturbation theory (a common technique in ∞-theories) and *representations up to (strong) homotopy (RUTHs).*

L_∞ and physics

This suggested an L_∞-algebra and module, but things were not so simple.

My student Lars Kjeseth in his UNC thesis Kjeseth (2001a,b) showed that the appropriate structure was that of a *strong homotopy Lie-Rinehart algebra* (shLR algebra).

This concept then lay dormant until resurrected around 2013 in the work of Johannes Huebschmann and of Luca Vitagliano.

Let's return to Geometric Combinatorics.

Geometric combinatorics

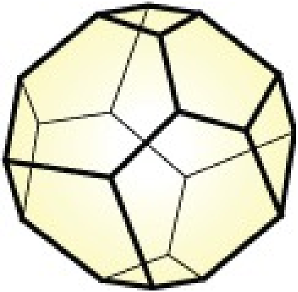
The initial application of the associahedra did not stop with those particular polytopes. For A_n-maps (morphisms up to homotopy) of A_n-spaces, I needed *multiplihedra*. Here the related algebra was straight forward, again generalizing Sugawara, but realization as a convex polytope took a lot longer. I think Stefan was first.

Of course, the *permutahedra* (or permutohedra) were already well known. According to Ziegler (1995), permutahedra were first studied by Schoute (1911) !! It can easily be realized as the convex hull of the points given by pemutations of the coordinates of the point $(0, 1, \ldots, n) \in \mathbb{R}^n$,

The associahedra and permutahedra were combined in the *permutoassociahedron* by Kapranov (1993), which had applications to asymptotic zones for the KZ equation.

Geometric combinatorics

To me, the next most striking were the *cyclohedra* found in Bott-Taubes self-linking invariants for knots (1994). Again these were related to compactification of configuration spaces. In this case, configurations of distinct points on the circle S^1. For n points in S^1, the compactification is written as $S^1 \times W_n$ where W_n is the cyclohedron. W_3 is a hexagon and here for n=4:



Higher and higher

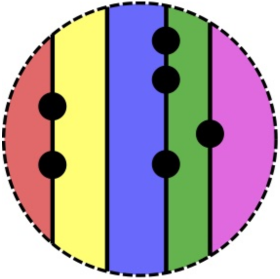
As $\infty-$ was prefixed to more and more concepts, they were subsumed in the now fairly ubiquitous term *higher structures*, even extensively in mathematical physics.

These include extensions of classical algebra/category theory to the differential graded context.

Fukaya (1995)'s notion of an A_∞-category related to Morse theory is among the most productive.

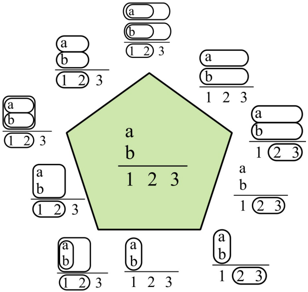
Higher and higher

A very recent example due to Nate Bottman circles back to the associahedra and also to Sasha Voronov's Homotopy Gerstenhaber Algebras (in fact, sh Gerstenhaber algebras). They both consider configurations of beaded lines: r vertical lines in $\mathbb{R}^2$ with x-positions $x_1, \ldots, x_r$, along with n_i marked points on the i-th line with y-positions $y_{i1}, \ldots, y_{in_i}$, up to an appropriate equivalence.



Higher and higher

Bottman then constructs the *2-associahedra* in terms of *2-bracketings* which include the associahedra.



See Bottman (2017) for 3-dimensional models.

The shapes of things to come?

∞-associahedra?

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