MORE FRIENDS FOR
THE ASSOCIAHEDRON

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(CNRS & LIX)

Geometry
and Combinatorics
of Associativity
October 23–27, 2017
PROGRAM

0. VARIOUS ASSOCIAHEDRA
   • 3 constructions
   • Loday’s associahedron
   • The Loday-Ronco Hopf algebra

I. PERMUTREEHEDRA
   • Permutrees and permutree lattices
   • Permutreehedra
   • The permutree Hopf algebra

II. QUOTIENTTOPES
   • Lattice quotients and arc diagrams
   • Quotienttopes
   • Hopf algebras on arc diagrams

III. THE UNIVERSAL ASSOCIAHEDRON AND ITS PROJECTIONS
   • g- and c-vectors
   • The g-vector fan is polytopal
   • The universal associahedron
   • Sections and projections
   • Extensions to cluster algebras

IV. NON-KISSING COMPLEXES AND GENTLE ASSOCIAHEDRA
   • Non-kissing complexes
   • Gentle associahedra
   • Non-kissing lattice
0. VARIOUS ASSOCIAHEDRA
FANS & POLYTOPES

Ziegler, Lectures on polytopes (’95)
Matoušek, Lectures on Discrete Geometry (’02)
**Simplicial Complex**

A simplicial complex is a collection of subsets of $X$ downward closed.

Example:

$$X = [n] \cup [n]$$

$$\Delta = \{ I \subseteq X \mid \forall i \in [n], \{i, i\} \not\subseteq I \}$$
FANS

polyhedral cone = positive span of a finite set of $\mathbb{R}^d$
  = intersection of finitely many linear half-spaces
fan = collection of polyhedral cones closed by faces
  and where any two cones intersect along a face

simplicial fan = maximal cones generated by $d$ rays
polytope = convex hull of a finite set of $\mathbb{R}^d$
    = bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations

simple polytope = facets in general position = each vertex incident to $d$ facets
$P$ polytope, $F$ face of $P$

normal cone of $F =$ positive span of the outer normal vectors of the facets containing $F$

normal fan of $P =$ \{ normal cone of $F$ | $F$ face of $P$ \}

simple polytope $\implies$ simplicial fan $\implies$ simplicial complex
EXM: PERMUTAHEDRON

Hohlweg, *Permutahedra and associahedra* ('12)
Permutohedron $\text{Perm}(n)$

$= \text{conv} \{ (\sigma(1), \ldots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1} \}$

$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{j \in J} x_j \geq \left( |J| + 1 \right)^2 \right\}$

connections to
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group
Coxeter fan

= fan defined by the hyperplane arrangement
\[ \{ x \in \mathbb{R}^{n+1} \mid x_i = x_j \}_{1 \leq i < j \leq n+1} \]
= collection of all cones
\[ \{ x \in \mathbb{R}^{n+1} \mid x_i < x_j \text{ if } \pi(i) < \pi(j) \} \]
for all surjections \( \pi : [n + 1] \to [n + 1 - k] \)
ASSOCIAHEDRA

Ceballos-Santos-Ziegler,
Many non-equivalent realizations of the associahedron (’11)
Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion.

vertices ↔ triangulations
edges ↔ flips
faces ↔ dissections

vertices ↔ binary trees
edges ↔ rotations
faces ↔ Schröder trees
**VARIOUS ASSOCIAHEDRA**

**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion.

Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE

- Gelfand-Kapranov-Zelevinsky (’94)
- Billera-Filliman-Sturmfels (’90)

LODAY’S ASSOCIAHEDRON

- Loday (’04)
- Hohlweg-Lange (’07)
- Hohlweg-Lange-Thomas (’12)

CHAP.-FOM.-ZEL.’S ASSOCIAHEDRON

- Chapoton-Fomin-Zelevinsky (’02)
- Ceballos-Santos-Ziegler (’11)

(Pictures by CFZ)
### THREE FAMILIES OF REALIZATIONS

<table>
<thead>
<tr>
<th>SECONDARY POLYTOPE</th>
<th>LODAY’S ASSOCIAHEDRON</th>
<th>CHAP.-FOM.-ZEL.’S ASSOCIAHEDRON</th>
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<tbody>
<tr>
<td><img src="image1" alt="Gelfand-Kapranov-Zelevinsky ('94)" /> Billera-Fillman-Sturmfels ('90)</td>
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<td><img src="image3" alt="Chapoton-Fomin-Zelevinsky ('02) Ceballos-Santos-Ziegler ('11)" /></td>
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</tbody>
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- **Hopf algebra**
- **Cluster algebras**

(Pictures by CFZ)
LODAY’S ASSOCIATEDRON

Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)
Loday, *Realization of the Stasheff polytope* ('04)
LODAY’S ASSOCIAHEDRON

\[ \text{Asso}(n) := \text{conv} \{ \mathbf{L}(T) \mid T \text{ binary tree} \} = \mathbb{H} \cap \bigcap_{1 \leq i \leq j \leq n+1} \mathbf{H}^\geq(i, j) \]

\[ \mathbf{L}(T) := \left[ \ell(T, i) \cdot r(T, i) \right]_{i \in [n+1]} \quad \mathbf{H}^\geq(i, j) := \left\{ x \in \mathbb{R}^{n+1} \left| \sum_{i \leq k \leq j} x_i \geq \binom{j - i + 2}{2} \right. \right\} \]

Shnider-Sternberg, Quantum groups: From coalgebras to Drinfeld algebras ('93)

Loday, Realization of the Stasheff polytope ('04)
Loday’s Associahedron

\[ \text{Asso}(n) := \text{conv} \{ \mathbf{L}(T) \mid T \text{ binary tree} \} = \mathbb{H} \cap \bigcap_{1 \leq i \leq j \leq n+1} \mathbf{H}^{\geq}(i, j) \]

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Shnider-Sternberg, *Quantum groups: From coalgebras to Drinfeld algebras* ('93)

Loday, *Realization of the Stasheff polytope* ('04)
Loday’s Association
Tamari lattice = slope increasing flips on triangulations
Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees

Tamari Festschrift ('12)
Tamari lattice  = slope increasing flips on triangulations
= right rotations on binary trees
= lattice quotient of the weak order by the sylvester congruence
Tamari lattice = slope increasing flips on triangulations
= right rotations on binary trees
= lattice quotient of the weak order by the sylvester congruence
= orientation of the graph of the associahedron in direction $e \to w_0$. 
Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)
Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
For $n, n' \in \mathbb{N}$, consider the set of perms of $\mathcal{S}_{n+n'}$ with at most one descent, at position $n$:

$$\mathcal{G}^{(n,n')} := \{ \tau \in \mathcal{S}_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n') \}$$

For $\tau \in \mathcal{S}_n$ and $\tau' \in \mathcal{S}_{n'}$, define

- **shifted concatenation** $\tau \bar{\tau}' = [\tau(1), \ldots, \tau(n), \tau'(1) + n, \ldots, \tau'(n') + n] \in \mathcal{G}_{n+n'}$
- **shifted shuffle product** $\tau \shuffle \tau' = \{ (\tau \bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathcal{G}^{(n,n')} \} \subset \mathcal{G}_{n+n'}$
- **convolution product** $\tau \star \tau' = \{ \pi \circ (\tau \bar{\tau}') \mid \pi \in \mathcal{G}^{(n,n')} \} \subset \mathcal{G}_{n+n'}$

**Exm:**

- $12 \bar{\shuffle} 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$
- $12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$
Combinatorial Hopf Algebra = combinatorial vector space $\mathcal{B}$ endowed with

- **product** $\cdot : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$
- **coproduct** $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$

which are "compatible", ie.

$$
\begin{align*}
\mathcal{B} \otimes \mathcal{B} & \quad \cdot \quad \mathcal{B} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & \quad I \otimes \text{swap} \otimes I \quad \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}
\end{align*}
$$

**THM.** The vector space $k\mathcal{G} = \bigoplus_{n \in \mathbb{N}} k\mathcal{G}_n$ with basis $(F_\tau)_{\tau \in \mathcal{G}}$ endowed with

$$
F_\tau \cdot F_{\tau'} = \sum_{\sigma \in \tau \uplus \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \star \tau'} F_\tau \otimes F_{\tau'}
$$

is a combinatorial Hopf algebra.

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)
**THM.** The vector space $k\mathcal{S} = \bigoplus_{n \in \mathbb{N}} k\mathcal{S}_n$ with basis $(F_\tau)_{\tau \in \mathcal{S}}$ endowed with

$$F_\tau \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \ast \tau'} F_\tau \otimes F_{\tau'}$$

is a combinatorial Hopf algebra.

Malvenuto-Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra* ('95)

**THM.** For a binary search tree $T$, consider the element $P_T := \sum_{\tau \in \mathcal{S} \mid \text{BST}(\tau) = T} F_\tau = \sum_{\tau \in \mathcal{L}(T)} F_\tau$.

These elements generate a Hopf subalgebra $k\mathcal{T}$ of $k\mathcal{S}$.

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)

binary search tree insertion of 2751346
I. PERMUTREEHEDRA

Chatel-P., *Cambrian Hopf Algebras* ('17)

P.-Pons, *Permutrees* ('17)
PERMUTREES

Chatel-P., *Cambrian Hopf Algebras* (’17)
P.-Pons, *Permutrees* (’17)
permutree = directed (bottom to top) and labeled (bijectively by \([n]\)) tree such that

increasing tree = directed and labeled tree such that labels increase along arcs

leveled permutree = directed tree with a permutree labeling and an increasing labeling
<table>
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<th>Examples.</th>
<th>decoration</th>
<th>permutrees</th>
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<tbody>
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<td>$\mathbb{D}^n$</td>
<td>$\leftrightarrow$</td>
<td>permutations of $[n]$</td>
</tr>
<tr>
<td>$\mathbb{H}^n$</td>
<td>$\leftrightarrow$</td>
<td>standard binary search trees</td>
</tr>
<tr>
<td>${\mathbb{D}, \mathbb{H}}^n$</td>
<td>$\leftrightarrow$</td>
<td>Cambrian trees</td>
</tr>
<tr>
<td>$\otimes^n$</td>
<td>$\leftrightarrow$</td>
<td>binary sequences</td>
</tr>
</tbody>
</table>

![Diagram](https://via.placeholder.com/150)
permutree correspondence = decorated permutation \(\mapsto\) leveled permutree

Exm: decorated permutation \(\overline{2751346}\)

Reading, Cambrian lattices (’06)
Lange-P., Associahedra via spines (’13+)
Chatel-P., Cambrian Hopf algebras (’17)
P.-Pons, Permutrees (’17)
permutree correspondence = decorated permutation $\mapsto$ leveled permutree

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permute correspondence $=$ decorated permutation $\mapsto$ leveled permutree

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PROP. bijection decorated permutation $\leftrightarrow$ leveled permutree.

Reading, *Cambrian lattices* ('06)
Lange-P., *Associahedra via spines* ('13+)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)
permutree correspondence = decorated permutation \rightarrow \text{leveled permutree}

Exm: decorated permutation \overline{2751346}

\[
P(\tau) = \text{P-symbol of } \tau = \text{permutree produced by permutree correspondence}
\]
\[
Q(\tau) = \text{Q-symbol of } \tau = \text{increasing tree produced by permutree correspondence}
\]

(analogy to Robinson-Schensted algorithm)
δ-permutree congruence = transitive closure of the rewriting rules

\[ UacVbW \equiv_\delta UcaVbW \quad \text{if} \quad a < b < c \quad \text{and} \quad \delta_b \in \{\otimes, \bigcirc\} \]
\[ UbVacW \equiv_\delta UbVcaW \quad \text{if} \quad a < b < c \quad \text{and} \quad \delta_b \in \{\otimes, \bigcirc\} \]

where \( a, b, c \) are elements of \([n]\) while \( U, V, W \) are words on \([n]\)

\[
\text{PROP.} \quad \tau \equiv_\delta \tau' \iff P(\tau) = P(\tau').
\]
\( \delta \)-permutree congruence = transitive closure of the rewriting rules

\[
U acV bW \equiv_\delta U caV bW \quad \text{if} \ a < b < c \ \text{and} \ \delta_b \in \{\ominus, \otimes\}
\]
\[
U bV acW \equiv_\delta U bV caW \quad \text{if} \ a < b < c \ \text{and} \ \delta_b \in \{\oplus, \otimes\}
\]

where \( a, b, c \) are elements of \([n]\) while \( U, V, W \) are words on \([n]\)

**PROP.** \( \tau \equiv_\delta \tau' \iff P(\tau) = P(\tau') \).

**PROP.** The permutree congruence class labeled by permutree \( T \) is given by

\[
\{ \tau \in \mathcal{G}^\delta \mid P(\tau) = T \} = \{ \text{linear extensions of} \ T \}. 
\]

**PROP.** The permutree classes are intervals of the weak order.

Minimums avoid \( b - ca \) with \( \delta_b \in \{\ominus, \otimes\} \) and \( ca - b \) with \( \delta_b \in \{\oplus, \otimes\} \).

Maximums avoid \( b - ac \) with \( \delta_b \in \{\oplus, \otimes\} \) and \( ac - b \) with \( \delta_b \in \{\ominus, \otimes\} \).

Reading, Cambrian lattices ('06)  
P.-Pons, Permutrees ('17)
Rotation operation preserves permutrees:

increasing rotation = rotation of edge $i \rightarrow j$ where $i < j$

PROP. The transitive closure of the increasing rotation graph is the permutree lattice. $P$ defines a lattice homomorphism from the weak order to the permutree lattice.

Reading, *Cambrian lattices* (’06)
P.-Pons, *Permutrees* (’17)
\[\delta \text{ refines } \delta' \text{ when } \delta_i \preceq \delta'_i \text{ for all } i \in [n] \text{ for the order } \bigodot \preceq \bigodot, \bigodot \preceq \bigotimes.\]

**PROP.** When \(\delta\) refines \(\delta'\), the \(\delta\)-permutree congruence classes refine the \(\delta\)-permutree congruence classes: \(\sigma \equiv_{\delta} \tau \implies \sigma \equiv_{\delta'} \tau.\)

It defines a surjection \(\Psi_{\delta}^{\delta'}\) from the \(\delta\)-permutrees to the \(\delta'\)-permutrees.
PERMUTREEHEDRA

Loday, *Realization of the Stasheff polytope* (’04)
Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* (’07)
Lange-P., *Using spines to revisit a construction of the associahedron* (’15)
Chatel-P., *Cambrian Hopf Algebras* (’17)
P.-Pons, *Permutrees* (’17)
PERMUTREE FAN

For a permutree $T$, define

$$C^\diamond(T) := \{ x \in \mathbb{R}^n \mid x_i \leq x_j \text{ for any } i \rightarrow j \text{ in } T \}$$

$$= \mathbb{I} + \text{cone} \left\{ \sum_{j \in J} |I| e_j - \sum_{i \in I} |J| e_i \right\} \text{ for all edge cuts } (I \parallel J) \text{ in } T \}

THM. For any $\delta \in \{\bigcirc, \bigotimes, \bigotimes, \bigotimes\}^n$, the collection of cones $\{C^\diamond(T) \mid T \text{ $\delta$-permutree}\}$ together with all their faces define a complete simplicial fan, the $\delta$-permutree fan $F(\delta)$.

P.-Pons, Permutrees ('17)

Examples.

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<tr>
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<th>permutrees</th>
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</thead>
<tbody>
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<td>$\bigcirc^n$</td>
<td>braid fan</td>
</tr>
<tr>
<td>$\bigotimes^n$</td>
<td>binary tree fan</td>
</tr>
<tr>
<td>${\bigotimes, \bigotimes}^n$</td>
<td>Cambrian fan</td>
</tr>
<tr>
<td>$\bigotimes^n$</td>
<td>fan of the arrangement ${x_i = x_{i+1} \mid i \in [n-1]}$</td>
</tr>
</tbody>
</table>
The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{P}T(\delta)$, defined equivalently as

(i) the convex hull of the points

$$a(T)_i = \begin{cases} 
    d + 1 & \text{if } \delta_i = \emptyset, \\
    d + 1 + \ell_r & \text{if } \delta_i = \bigcirc, \\
    d + 1 - \bar{\ell}_r & \text{if } \delta_i = \bigotimes, \\
    d + 1 + \ell_r - \bar{\ell}_r & \text{if } \delta_i = \bigotimes,
\end{cases}$$

for all $\delta$-permutrees $T$,

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

$$\mathbb{H}^\geq(I) := \left\{ x \in \mathbb{R}^n \left| \sum_{i \in I} x_i \geq \left( |I| + 1 \right) \right. \right\}$$

for all edge cuts $(I \parallel J)$ of all $\delta$-permutrees.

P.-Pons, Permutrees (’17)
**THM.** The permutree fan $\mathcal{F}(\delta)$ is the normal fan of the permutreehedron $\mathbb{PT}(\delta)$. 
PROPOSITION. \( U := (n, n - 1, \ldots, 2, 1) - (1, 2, \ldots, n - 1, n) = \sum_{i \in [n]} (n + 1 - 2i) e_i \)

graph of \( PT(\delta) \) oriented by \( U = \) Hasse diagram of the \( \delta \)-permutree lattice.
PROP. refinement $\delta \preceq \delta'$ $\implies$ inclusion $\text{PT}(\delta) \subset \text{PT}(\delta')$. 
MATRIOCHKA PERMUTTREEHEDRA
MATRIOCHKA PERMUTREEHEDRA
PERMUTREE ALGEBRA

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Hivert-Novelli-Thibon, *The algebra of binary search trees* ('05)
Chatel-P., *Cambrian Hopf Algebras* ('17)
P.-Pons, *Permutrees* ('17)
For decorated permutations:

- decorations are attached to values in the shuffle
- decorations are attached to positions in the convolution

Exm: $\begin{bmatrix} 1 & 2 \\ \bar{2} & 3 \end{bmatrix} = \begin{bmatrix} 12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ \bar{2} & 3 \end{bmatrix} \star \begin{bmatrix} 23 \end{bmatrix} = \begin{bmatrix} 12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231 \end{bmatrix}$

\[ k\mathcal{G}_{\{\circ,\circ,\circ,\circ\}} = \text{Hopf algebra with basis } (F_{\tau})_{\tau \in \mathcal{G}_{\{\circ,\circ,\circ,\circ\}}} \text{ and where} \]

\[ F_{\tau} \cdot F_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} F_{\sigma} \quad \text{and} \quad \Delta F_{\sigma} = \sum_{\sigma \in \tau \star \tau'} F_{\tau} \otimes F_{\tau'} \]
Permutree algebra $= \text{vector subspace } k \mathcal{PT} \text{ of } k \mathcal{G}\{\emptyset, \circ, \bullet, \star\} \text{ generated by }$

$$P_T := \sum_{\tau \in \mathcal{G}\{\emptyset, \circ, \bullet, \star\} : P(\tau) = T} F_\tau = \sum_{\tau \in \mathcal{L}(T)} F_\tau,$$

for all permutrees $T$.

Exm: $P = F_{2135476} + F_{2135746} + F_{2137546} + \cdots + F_{7523146} + F_{7523416} + F_{7523461}$

**Theo.** $k \mathcal{PT}$ is a subalgebra of $k \mathcal{G}\{\emptyset, \circ, \bullet, \star\}$.

Loday-Ronco, *Hopf algebra of the planar binary trees* ('98)
Hivert-Novelli-Thibon, *The algebra of binary search trees* ('05)
Chatel-P., *Cambrian Hopf algebras* ('17)
P.-Pons, *Permutrees* ('17)

**GAME:** Explain the product and coproduct directly on the permutrees...
**PRODUCT IN PERMUTREE ALGEBRA**

\[
P_T \cdot P_T' = F_{12} \cdot (F_{213} + F_{231})
\]

\[
= \left( F_{12435} + F_{12453} + F_{14235} + F_{41235} + F_{41253} + F_{41523} \right) + \left( F_{14325} + F_{14352} + F_{41325} + F_{41352} + F_{45132} \right) + \left( F_{43125} + F_{43152} + F_{43512} + F_{45312} \right)
\]

\[
= \text{PROP.}
\]

For any permutrees $T$ and $T'$,

\[
P_T \cdot P_T' = \sum_S P_S
\]

where $S$ runs over the interval $[T \nearrow \overline{T'}, T \searrow \overline{T'}]$ in the $\delta(T)\delta(T')$-permutree lattice.
\[ \Delta P = \Delta (F_{213} + F_{231}) \]
\[ = 1 \otimes (F_{213} + F_{231}) + F_I \otimes F_{12} + F_I \otimes F_{21} + F_{21} \otimes F_I + F_{12} \otimes F_1 + (F_{213} + F_{231}) \otimes 1 \]
\[ = 1 \otimes P + P P \otimes P + P P \otimes P + P P \otimes P + P P \otimes P + P P \otimes 1 \]
\[ = 1 \otimes P + P P \otimes (P \cdot P) + P P \otimes P + P P \otimes P + P P \otimes 1. \]

**PROP.** For any permutree \( S \),

\[ \Delta \mathcal{P}_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} \mathcal{P}_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} \mathcal{P}_{T'} \right) \]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the forests above and below \( \gamma \) respectively.
\[ \Delta P = \Delta (F_{213} + F_{231}) \]

\[ = 1 \otimes (F_{213} + F_{231}) + F_I \otimes F_{12} + F_I \otimes F_{21} + F_{21} \otimes F_I + F_{12} \otimes F_1 + (F_{213} + F_{231}) \otimes 1 \]

\[ = 1 \otimes P + P_1 \otimes P + P_2 \otimes P + P_3 \otimes P + P_4 \otimes P + P_5 \otimes P + P_6 \otimes P \]

\[ = 1 \otimes P + P_1 \otimes (P_2 \cdot P_3) + P_4 \otimes P_5 + P_6 \otimes P + P_7 \otimes 1. \]

**PROP.** For any permutree \( S \),

\[ \Delta P_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} P_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} P_{T'} \right) \]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the forests above and below \( \gamma \) respectively.
EXTENSIONS

• Schröder permutrees
EXTENSIONS

• **Schröder permutrees**

• **arbitrary finite Coxeter groups**
  somewhere between the $W$-permutahedron and the $W$-associahedron

Reading, *Cambrian lattices* ('06)
Reading-Speyer, *Cambrian fans* ('09)
Hohlweg-Lange-Thomas, *Permutahedra and generalized associahedra* ('11)
P.-Stump, *Brick polytopes of spherical subword complexes & gen. assoc.* ('15)
Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
II. QUOTIENTOPES

P.-Santos, *Quotientopes* ('17+)
P., *Hopf algebras on decorated noncrossing arc diagrams* ('17+)
LATTICE SETUP

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
Reading, *Noncrossing arc diagrams and canonical join representations* ('15)
Reading, *Finite Coxeter groups and the weak order* ('16)
Reading, *Lattice theory of the poset of regions* ('16)
lattice = poset \((L, \leq)\) with a meet \(\land\) and a join \(\lor\)

join representation of \(x \in L\) = subset \(J \subseteq L\) such that \(x = \bigvee J\).

\(x = \bigvee J\) irredundant if \(\not\exists J' \subsetneq J\) with \(x = \bigvee J'\)

JR are ordered by containment of order ideals: \(J \leq J' \iff \forall y \in J, \exists y' \in J', y \leq y'\)

canonical join representation of \(x\) = minimal irred. join representation of \(x\) (if it exists)

\(\Rightarrow \) “lowest way to write \(x\) as a join”
\( \sigma \) permutation
inversions of \( \sigma = \) pair \((\sigma_i, \sigma_j)\) such that \( i < j \) and \( \sigma_i > \sigma_j \)
weak order = permutations of \( S_n \) ordered by inclusion of inversion sets
σ permutation

Inversions of \( \sigma = \) pair \((\sigma_i, \sigma_j)\) such that \(i < j\) and \(\sigma_i > \sigma_j\)

Weak order = permutations of \(\mathcal{S}_n\) ordered by inclusion of inversion sets

Descent of \(\sigma = \) \(i\) such that \(\sigma_i > \sigma_{i+1}\)
\(\sigma\) permutation

inversions of \(\sigma = \text{pair} (\sigma_i, \sigma_j)\) such that \(i < j\) and \(\sigma_i > \sigma_j\)

weak order = permutations of \(S_n\) ordered by inclusion of inversion sets

descent of \(\sigma = i\) such that \(\sigma_i > \sigma_{i+1}\)

join-irreducible \(\lambda(\sigma, i)\)

---

THM. Canonical join representation of \(\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)\).

Reading, Noncrossing arc diagrams and canonical join representations (’15)
THM. Canonical join representation of $\sigma = \bigvee_{\sigma_i > \sigma_{i+1}} \lambda(\sigma, i)$.

Reading, *Noncrossing arc diagrams and canonical join representations* (’15)
\[ \sigma = (a, b, n, S) \text{ with } 1 \leq a < b \leq n \text{ and } S \subseteq ]a, b[ \]
\( \sigma = 2537146 \)

draw the table of points \((\sigma_i, i)\)

draw all arcs \((\sigma_i, i) \rightarrow (\sigma_{i+1}, i+1)\) with

descents in red and ascent in green

project down the red arcs and up the green arcs

allowing arcs to bend but not to cross or pass points

\( \delta(\sigma) = \) projected red arcs

\( \delta(\sigma) = \) projected green arcs

noncrossing arc diagrams = set \( \mathcal{D} \) of arcs st. \( \forall \alpha, \beta \in \mathcal{D} \):

- \( \text{left}(\alpha) \neq \text{left}(\beta) \) and \( \text{right}(\alpha) \neq \text{right}(\beta) \),
- \( \alpha \) and \( \beta \) are not crossing.

**THM.** \( \sigma \rightarrow \delta(\sigma) \) and \( \sigma \rightarrow \delta(\sigma) \) are bijections from permutations to noncrossing arc diagrams.

Reading, *Noncrossing arc diagrams and can. join representations* (’15)
LATTICE CONGRUENCES

lattice congruence = equiv. rel. \( \equiv \) on \( L \) which respects meets and joins

\[ x \equiv x' \quad \text{and} \quad y \equiv y' \implies x \land y \equiv x' \land y' \quad \text{and} \quad x \lor y \equiv x' \lor y' \]

lattice quotient of \( L/\equiv \) = lattice on equiv. classes of \( L \) under \( \equiv \) where

- \( X \leq Y \iff \exists x \in X, y \in Y, x \leq y \)
- \( X \land Y \) = equiv. class of \( x \land y \) for any \( x \in X \) and \( y \in Y \)
- \( X \lor Y \) = equiv. class of \( x \lor y \) for any \( x \in X \) and \( y \in Y \)
binary search tree insertion of 2751346
EXM: TAMARI LATTICE AS LATTICE QUOTIENT OF WEAK ORDER

binary search tree insertion of 2751346

4321
3421
3241
2431
2341
2314
3124
3142
3214
2134
1324
1234
1342
1423
1432
4132
4213
4231
4312
3412
3142
2413
2143
1423
1342
1432
2341
1243
1234
2134
3124
1324
3214
2341
1234
2314
3142
2431
3241
4321

LATTICE QUOTIENTS AND CANONICAL JOIN REPRESENTATIONS

≡ lattice congruence on $L$, then

- each class $X$ is an interval $[\pi_\downarrow(X), \pi_\uparrow(X)]$
- $L/\equiv$ is isomorphic to $\pi_\downarrow(L)$ (as poset)
- canonical join representations in $L/\equiv$ are canonical join representations in $L$ that do not involve join irreducibles $x$ with $\pi_\downarrow(x) \neq x$.

**THM.** $\equiv$ lattice congruence of the weak order on $\mathcal{S}_n$

Let $\mathcal{I}_\equiv = \text{arcs corresponding to join irreducibles } \sigma$ with $\pi_\downarrow(\sigma) = \sigma$

- $\pi_\downarrow(\sigma) = \sigma \iff \sigma$ has no descent $i$ st. $\alpha(\sigma, i) \notin \mathcal{I}_\equiv$.
- the map $\mathcal{S}_n/\equiv \to \{\text{nc arc diagrams in } \mathcal{I}_\equiv\}$ is a bijection.
- $\equiv$ is the transitive closure of the rewriting rule $\sigma \to \sigma \cdot (i \ i + 1)$ where $i$ descent of $\sigma$ such that $\alpha(\sigma, i) \notin \mathcal{I}_\equiv$.

Reading, Noncrossing arc diagrams and can. join representations ('15)
THM. $\mathcal{I}_\equiv = \text{arcs corresponding to join irreducibles } \sigma \text{ with } \pi_\downarrow(\sigma) = \sigma$.

Bijection $\mathfrak{S}_n/\equiv \leftrightarrow \{\text{nc arc diagrams in } \mathcal{I}_\equiv\}$.

What sets of arcs can be $\mathcal{I}_\equiv$?

$(a, d, n, S)$ forces $(b, c, n, T)$ when $a \leq b < c \leq d$ and $T = S \cap ]b, c[$.

THM. $\mathcal{I}$ set of arcs. $\exists$ lattice cong. $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{I} = \mathcal{I}_\equiv \iff \mathcal{I}$ closed by forcing.

Reading, Noncrossing arc diagrams and can. join representations ('15)
arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing
arc ideal = ideal of the forcing poset on arcs = subsets of arcs closed by forcing

fix $k \geq 0$ and some red walls above, below and in between the points allow arcs that cross at most $k$ walls

weak order  Tamari lattice  diagonal rectangulations  Cambrian lattices  $k$-twist lattices
\( \mathcal{I}_\equiv \) closed by forcing

bijection \( \mathcal{G}_n/\equiv \rightarrow \{ \text{nc arc diagrams in } \mathcal{I}_\equiv \} \)

\[ X \mapsto \delta(\pi_\downarrow(X)) \]
$I_{\equiv}$ closed by forcing surjection $\mathfrak{S}_n \longrightarrow \{\text{nc arc diagrams in } I_{\equiv}\}$
$\sigma \longmapsto \delta(\pi_\downarrow(\sigma))$
\(\mathcal{I}_\equiv\) closed by forcing

\[
\text{surjection } \mathfrak{S}_n \longrightarrow \{\text{nc arc diagrams in } \mathcal{I}_\equiv\}
\]

\(\sigma \mapsto \delta(\pi^\downarrow(\sigma))\)

\(\Box(\sigma) = \{(i, j) \mid 1 \leq i < j \leq n, \sigma_i > \sigma_j \text{ and } \sigma([i, j]) \cap ]\sigma_j, \sigma_i[ = \emptyset\}\)

ordered by \((i, j) \prec (k, \ell) \iff i \leq k < \ell \leq j \text{ and } \sigma_k \geq \sigma_j > \sigma_i \geq \sigma_\ell\)

\(\alpha(i, j, \sigma) = (\sigma_j, \sigma_i, n, \{\sigma_k \mid j < k \text{ and } \sigma_k \in ]\sigma_j, \sigma_i[\}\})\)

\(\Box_{\mathcal{I}_\equiv}(\sigma) = \prec\text{-maximal elem. in } \{(i, j) \in \Box(\sigma) \mid \alpha(i, j, \sigma) \in \mathcal{I}_\equiv\}\)

**PROP.** \(\delta(\pi^\downarrow(\sigma)) = \{\alpha(i, j, \sigma) \mid (i, j) \in \Box_{\mathcal{I}_\equiv}(\sigma)\}.\)
FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS AGAIN

binary trees  diagonal quadrangulations  permutrees  $k$-twists
QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
P.-Santos, *Quotientopes* ('17+)
arcs decompose the hyperplanes \( \{ x \in \mathbb{R}^n \mid x_i = x_j \} \) of the braid arrangement into shards

\[
\text{shard } \Sigma(i, j, n, S) := \left\{ x \in \mathbb{R}^n \mid x_i = x_j \text{ and } \begin{bmatrix} x_i \leq x_k & \text{for all } k \in S \text{ while } \\ x_i \geq x_k & \text{for all } k \in ]i, j[ \setminus S \end{bmatrix} \right\}
\]
arcs decompose the hyperplanes \( \{x \in \mathbb{R}^n \mid x_i = x_j\} \) of the braid arrangement into shards

\[
\text{shard } \Sigma(i, j, n, S) := \left\{ x \in \mathbb{R}^n \mid x_i = x_j \text{ and } \begin{cases} x_i \leq x_k \text{ for all } k \in S \text{ while } \\ x_i \geq x_k \text{ for all } k \in ]i, j[ \setminus S \end{cases} \right\}
\]

**THM.** \( \equiv \) lattice congruence on \( \mathcal{S}_n \) with arcs \( \mathcal{I}_\equiv \)

The collection of cones defined equiv. as

- the cones obtained by glueing the Coxeter regions of the permutations in the same congruence class of \( \equiv \)
- the complements of the union of the shards \( \Sigma(\alpha) \) for all arcs \( \alpha \in \mathcal{I}_\equiv \)

forms a fan \( \mathcal{F}_\equiv \) of \( \mathbb{R}^n \) whose dual graph realizes the lattice quotient \( \mathcal{S}_n/\equiv \).

Reading, *Lattice congruences, fans and Hopf algebras* (’05)
fix a forcing dominant function $f : \text{arcs} \to \mathbb{R}_{>0}$ i.e. st. $f(\alpha) > \sum_{\alpha' \succ \alpha} f(\alpha')$ for any arc $\alpha$.

for an arc $\alpha = (i, j, n, S)$ and a subset $\emptyset \neq R \subset [n]$ define the contribution

$$
\gamma(\alpha, R) := \begin{cases} 
1 & \text{if } |R \cap \{i, j\}| = 1 \text{ and } S = R \cap [i, j], \\
0 & \text{otherwise}
\end{cases}
$$

define height function $h$ for $\emptyset \neq R \subset [n]$ by $h^f(\equiv)(R) := \sum_{\alpha \in \equiv} f(\alpha) \gamma(\alpha, R)$.

**THM.** For any lattice congruence $\equiv$ on $\mathcal{S}_n$ and any forcing dominant function $f : \text{arcs} \to \mathbb{R}_{>0}$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the polytope

$$
P^f_\equiv := \left\{ x \in \mathbb{R}^n \mid \langle r(R) \mid x \rangle \leq h^f(\equiv)(R) \text{ for all } \emptyset \neq R \subset [n] \right\}.
$$

P.-Santos, Quotientopes (’17+)
QUOTIENTOPE LATTICE
QUOTIENTOPE LATTICE

POLYWOOD
outsidahedra
permutrees

insidahedra
quotientopes
\( \mathcal{H} \) hyperplane arrangement in \( \mathbb{R}^n \)

\( B \) distinguished region of \( \mathbb{R}^n \setminus \mathcal{H} \)

inversion set of a region \( C \) = set of hyperplanes of \( \mathcal{H} \) that separate \( B \) and \( C \)

poset of regions \( \text{Pos}(\mathcal{H}, B) = \) regions of \( \mathbb{R}^n \setminus \mathcal{H} \) ordered by inclusion of inversion sets

**THM.** The poset of regions \( \text{Pos}(\mathcal{H}, B) \)

- is never a lattice when \( B \) is not a simple region,
- is always a lattice when \( \mathcal{H} \) is a simplicial arrangement.

Björner-Edelman-Ziegler, *Hyperplane arrangements with a lattice of regions* ('90)

**THM.** If \( \text{Pos}(\mathcal{H}, B) \) is a lattice, and \( \equiv \) is a lattice congruence of \( \text{Pos}(\mathcal{H}, B) \), the cones obtained by glueing together the regions of \( \mathbb{R}^n \setminus \mathcal{H} \) in the same congruence class form a complete fan.

Reading, *Lattice congruences, fans and Hopf algebras* ('05)

Is the quotient fan polytopal?
HOPF ALGEBRAS ON ARC DIAGRAMS

P., Hopf algebras on decorated noncrossing arc diagrams (’17+)
DECORATED PERMUTATION

**decoration set** = a graded set \( \mathcal{X} := \bigsqcup_{n \geq 0} \mathcal{X}_n \) endowed with

- a **concatenation** \( \text{concat} : \mathcal{X}_m \times \mathcal{X}_n \rightarrow \mathcal{X}_{m+n} \)
- a **selection** \( \text{select} : \mathcal{X}_m \times \binom{[m]}{k} \rightarrow \mathcal{X}_k \)

such that

(i) \( \text{concat}(\mathcal{X}, \text{concat}(\mathcal{Y}, \mathcal{Z})) = \text{concat}(\text{concat}(\mathcal{X}, \mathcal{Y}), \mathcal{Z}) \)

(ii) \( \text{select}(\text{select}(\mathcal{X}, R), S) = \text{select}(\mathcal{X}, \{r_s \mid s \in S\}) \)

(iii) \( \text{concat}(\text{select}(\mathcal{X}, R), \text{select}(\mathcal{Y}, S)) = \text{select}(\text{concat}(\mathcal{X}, \mathcal{Y}), R \cup S^{\rightarrow m}) \)

where \( S^{\rightarrow m} := \{ s + m \mid s \in S \} \).

**Exm:**

- \( A^* = \) words on an alphabet \( A \), with concatenation and subwords
- labeled graphs, with shifted union and induced subgraphs
- ...
**DECORATED PERMUTATION**

**Decoration set** $\mathcal{X} := \bigcup_{n \geq 0} \mathcal{X}_n$ endowed with
- a **concatenation** $\text{concat} : \mathcal{X}_m \times \mathcal{X}_n \longrightarrow \mathcal{X}_{m+n}$
- a **selection** $\text{select} : \mathcal{X}_m \times \binom{[m]}{k} \longrightarrow \mathcal{X}_k$

such that

(i) $\text{concat}(\mathcal{X}, \text{concat}(\mathcal{Y}, \mathcal{Z})) = \text{concat}(\text{concat}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})$
(ii) $\text{select}(\text{select}(\mathcal{X}, R), S) = \text{select}(\mathcal{X}, \{r_s \mid s \in S\})$
(iii) $\text{concat}(\text{select}(\mathcal{X}, R), \text{select}(\mathcal{Y}, S)) = \text{select}(\text{concat}(\mathcal{X}, \mathcal{Y}), R \cup S \rightarrow m)$

where $S \rightarrow m := \{s + m \mid s \in S\}$.

**$\mathcal{X}$-decorated permutation** $= \text{pair} (\sigma, \mathcal{X})$ with $\sigma \in \mathcal{S}_n$ and $\mathcal{X} \in \mathcal{X}_n$.

**Standardization** $\text{std}((\rho, \mathcal{Z}), R) := \left(\text{stdp}(\rho, R), \text{select}(\mathcal{Z}, \rho^{-1}(R))\right)$

**THM.** The product $\cdot$ and coproduct $\triangle$ defined by

$$\begin{align*}
\mathbf{F}_{(\sigma, \mathcal{X})} \cdot \mathbf{F}_{(\tau, \mathcal{Y})} &:= \sum_{\rho \in \sigma \sqcup \tau} \mathbf{F}_{(\rho, \text{concat}(\mathcal{X}, \mathcal{Y}))} \\
\triangle \mathbf{F}_{(\rho, \mathcal{Z})} &:= \sum_{k=0}^{p} \mathbf{F}_{\text{std}((\rho, \mathcal{Z}), [k])} \otimes \mathbf{F}_{\text{std}((\rho, \mathcal{Z}), [p \setminus k])}
\end{align*}$$

endow the vector space of decorated permutations with a graded Hopf algebra structure.

*P., Hopf algebras on decorated noncrossing arc diagrams ('17+)*
a graded function $\Psi : \mathcal{X} = \bigsqcup_{n \geq 0} \mathcal{X}_n \longrightarrow \mathcal{I} = \bigsqcup_{n \geq 0} \mathcal{I}_n$ is conservative if

(i) $\Psi(\mathcal{X})^+ = \Psi(\mathcal{Y})^+$ are both subsets of $\Psi(\text{concat}(\mathcal{X}, \mathcal{Y}))$

(ii) $(r_a, r_b, p, S) \in \Psi(\mathcal{Z})$ implies $(a, b, q, \{c \mid r_c \in S\}) \in \Psi(\text{select}(\mathcal{Z}, R))$

$\mathcal{I}$ collection of arcs closed by forcing

surjection $\eta_{\mathcal{I}} : S_n \longrightarrow \{\text{nc arc diagrams in } \mathcal{I}\}$

$\sigma \mapsto \eta_{\mathcal{I}}(\sigma) = \delta(\pi_\downarrow(\sigma))$

$\mathcal{X}$-decorated noncrossing arc diagram $= (D, \mathcal{X})$ where $D$ is a non crossing arc diagram contained in $\Psi(\mathcal{X})$

**THM.** For a decorated noncrossing arc diagram $(D, \mathcal{X})$, define

$$P_{(D, \mathcal{X})} := \sum F(\sigma, \mathcal{X}),$$

where $\sigma$ ranges over the permutations such that $\eta_{\Psi(\mathcal{X})}(\sigma) = D$. The graded vector subspace $k\Theta := \bigoplus_{n \geq 0} k\Theta_n$ of $k\Psi$ generated by the elements $P_{(D, \mathcal{X})}$, for all $\mathcal{X}$-decorated noncrossing arc diagrams $(D, \mathcal{X})$, is a Hopf subalgebra of $k\Psi$. 

*P., Hopf algebras on decorated noncrossing arc diagrams ('17+)*
fix $k \geq 0$

$\mathcal{X} =$ words on $\mathbb{N}^4$ (each letter $a$ is made of 4 numbers $\ell_a + r_a$)

with $\text{concat}(a_1 \cdots a_m, b_1 \cdots b_n) = a_1 \cdots a_m b_1 \cdots b_n$

$\text{select}(c_1 \cdots c_p, R) = \bar{c}_{r_1} \cdots \bar{c}_{r_q}$ where $\bar{c}_{r_i} = \min_{r_{i-1} < k \leq r_i} \ell_{c_k} + \min_{r_i \leq k < r_{i+1}} d_{c_k}$

$\Psi(a_1 \cdots a_m) = \text{draw} \begin{vmatrix} u_{a_i} \\ d_{a_i} \\ \min(r_{a_i}, \ell_{a_{i+1}}) \end{vmatrix}$ red walls above $i$

allow arcs that cross at most $k$ walls

weak order Tamari lattice diagonal rectangulations Cambrian lattices $k$-twist lattices
extended arc = arc allowed to start at 0 or end at $n + 1$

$\mathcal{X} = \text{extended arc ideals with}$

concatenation:

selection:

$\Psi(\mathcal{X}) = \text{strict arcs in } \mathcal{X}$

$\Rightarrow$ Hopf algebra on all arc ideals containing the permutree algebra

P., Hopf algebras on decorated noncrossing arc diagrams ('17+)
III. THE UNIVERSAL ASSOCIAHEDRON AND ITS PROJECTIONS

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
Manneville-P., *Geometric realizations of the accordion complex* ('17+)
G- AND C-VECTORS
Consider simultaneously two \( n \)-gons:

- the red polygon supports a reference triangulation,
- the blue polygon is the ground set.
For $T_\circ$ red triangulation, $\delta_\circ \in T_\circ$ and $\delta_\bullet$ a blue diagonal, let

$$\varepsilon_\circ(\delta_\circ \in T_\circ, \delta_\bullet) = \begin{cases} 
1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in T_\circ \text{ as a } Z \\
-1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_\circ \in T_\circ \text{ as an } \Sigma \\
0 & \text{otherwise}
\end{cases}$$
For $T_o$ red triangulation, $\delta_o \in T_o$ and $\delta_*$ a blue diagonal, let

$$\varepsilon_o(\delta_o \in T_o, \delta_*) = \begin{cases} 
1 & \text{if $\delta_*$ slaloms on $\delta_o \in T_o$ as a } Z \\
-1 & \text{if $\delta_*$ slaloms on $\delta_o \in T_o$ as an } \Sigma \\
0 & \text{otherwise}
\end{cases}$$

$Z = 1$

$\Sigma = -1$

$\Psi = 0$
For $T_0$ red triangulation, $\delta_0 \in T_0$ and $\delta_\bullet$ a blue diagonal, let

$$
\varepsilon_\circ(\delta_0 \in T_0, \delta_\bullet) = \begin{cases} 
1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_0 \in T_0 \text{ as a } Z \\
-1 & \text{if } \delta_\bullet \text{ slaloms on } \delta_0 \in T_0 \text{ as an } \Sigma \\
0 & \text{otherwise}
\end{cases}
$$

$$
g(T_0, \delta_\bullet) = \text{g-vector of } \delta_\bullet \text{ with respect to } T_0 = \left[ \varepsilon_\circ(\delta_0 \in T_0, \delta_\bullet) \right]_{\delta_0 \in T_0} \in \mathbb{R}^{T_0}
$$

= alternating $\pm 1$ along the zigzag crossed by $\delta_\bullet$ in $T_0$

$$
g(T_0, (1_\bullet, 5_\bullet)) = e_{5\circ7_0} - e_{2\circ7_0} \quad g(T_0, (1_\bullet, 3_\bullet)) = -e_{2\circ4_0} \quad g(T_0, (5_\bullet, 7_\bullet)) = e_{5\circ7_0} \quad g(T_0, (3_\bullet, 5_\bullet)) = e_{5\circ7_0} - e_{4\circ7_0}
$$
G-VECTOR FAN

\[ g(T_\circ, \delta \cdot) = \text{g-vector of } \delta \cdot \text{ with respect to } T_\circ = \left[ \varepsilon_\circ(\delta \circ \in T_\circ, \delta \cdot) \right]_{\delta \circ \in T_\circ} \in \mathbb{R}^{T_\circ} \]

**THM.** For any red triangulation \( T_\circ \), the collection of cones

\[ \mathcal{F}_g(T_\circ) := \{ \mathbb{R}_{\geq 0}g(T_\circ, D \cdot) \mid D \cdot \text{ any blue dissection} \} \]

forms a complete simplicial fan, called \( g \)-vector fan of \( T_\circ \).
For $T_o$ red triangulation and $T_\bullet$ blue triangulation and two diagonals $\delta_o \in T_o$ and $\delta_\bullet \in T_\bullet$, let

$$\varepsilon_\bullet(\delta_o, \delta_\bullet \in T_\bullet) = \begin{cases} 
1 & \text{if } \delta_o \text{ slaloms on } \delta_\bullet \in T_\bullet \text{ as a } \Sigma \\
-1 & \text{if } \delta_o \text{ slaloms on } \delta_\bullet \in T_\bullet \text{ as an } Z \\
0 & \text{otherwise}
\end{cases}$$

$$c(T_o, \delta_\bullet \in T_\bullet) = \text{c-vector of } \delta_\bullet \text{ in } T_\bullet \text{ with respect to } T_o = \left[ \varepsilon_\bullet(\delta_o, \delta_\bullet \in T_\bullet) \right]_{\delta_o \in T_o} \in \mathbb{R}^{T_o}$$

$$= \pm \text{ charac. vector of diagonals of } T_o \text{ crossed by opposite neighbors of } \delta_\bullet$$

$$c(T_o, (1_\bullet, 5_\bullet) \in T_\bullet) = c(T_o, (1_\bullet, 3_\bullet) \in T_\bullet) = c(T_o, (5_\bullet, 7_\bullet) \in T_\bullet) = c(T_o, (5_\bullet, 7_\bullet) \in T_\bullet) =$$

$$-e_{2 \circ 7_\circ} \quad -e_{2 \circ 4_\circ} \quad e_{2_\circ 7_\circ} + e_{4_\circ 7_\circ} + e_{5_\circ 7_\circ} \quad -e_{4_\circ 7_\circ}$$
**G- AND C-VECTORS**

For $T_o$ red triangulation and $T_\bullet$ blue triangulation

$g(T_o, \delta_\bullet) = \text{g-vector of } \delta_\bullet \text{ with respect to } T_o = \left[ \varepsilon_o(\delta_o \in T_o, \delta_\bullet) \right]_{\delta_o \in T_o} \in \mathbb{R}^{T_o}$

$c(T_o, \delta_\bullet \in T_\bullet) = \text{c-vector of } \delta_\bullet \text{ in } T_\bullet \text{ with respect to } T_o = \left[ \varepsilon_\bullet(\delta_o, \delta_\bullet \in T_\bullet) \right]_{\delta_o \in T_o} \in \mathbb{R}^{T_o}$

\[\begin{align*}
g & = e_{5_\circ 7_\circ} - e_{2_\circ 7_\circ} \\
c & = -e_{2_\circ 7_\circ} - e_{2_\circ 4_\circ} + e_{5_\circ 7_\circ} - e_{4_\circ 7_\circ} + e_{4_\circ 7_\circ} + e_{5_\circ 7_\circ} + e_{5_\circ 7_\circ} - e_{4_\circ 7_\circ} + e_{5_\circ 7_\circ}
\end{align*}\]
**G- AND C-VECTORS**

For $T_o$ red triangulation and $T_\bullet$ blue triangulation

$$g(T_o, \delta_\bullet) = \text{g-vector of } \delta_\bullet \text{ with respect to } T_o = \left[ \varepsilon_o(\delta_o \in T_o, \delta_\bullet) \right]_{\delta_o \in T_o} \in \mathbb{R}^{T_o}$$

$$c(T_o, \delta_\bullet \in T_\bullet) = \text{c-vector of } \delta_\bullet \text{ in } T_\bullet \text{ with respect to } T_o = \left[ \varepsilon_\bullet(\delta_o, \delta_\bullet \in T_\bullet) \right]_{\delta_o \in T_o} \in \mathbb{R}^{T_o}$$

**PROP.** The g-vectors $g(T_o, T_\bullet)$ and the c-vectors $c(T_o, T_\bullet)$ form dual bases.

**PROP.** Duality: $g(T_o, T_\bullet) = -c(T_\bullet, T_o)^t$ and $c(T_o, T_\bullet) = -g(T_\bullet, T_o)^t$
ASSOCIAHEDRA FOR $G$-VECTOR FANS

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
$\mathbf{T}_o$-ZONOTOPE

$\mathbf{T}_o$-zonotope $= \text{Zono}(\mathbf{T}_o) = \text{Minkowski sum of all c-vectors } \mathbf{C}(\mathbf{T}_o) = \bigcup_{\mathbf{T}_1} \mathbf{c}(\mathbf{T}_o, \mathbf{T}_1)$

$$\text{Zono}(\mathbf{T}_o) = \sum_{\mathbf{c} \in \mathbf{C}(\mathbf{T}_o)} \mathbf{c}.$$ 

**PROP.** For any diagonal $\gamma_\circ$, $\text{Zono}(\mathbf{T}_o)$ has a facet defined by the inequality

$$\langle g(\mathbf{T}_o, \gamma_\circ) | \mathbf{x} \rangle \leq \omega(\gamma_\circ)$$

where $\omega(\gamma_\circ) = \text{number of red diagonals that cross } \gamma_\circ.$
Define
\[ p(T_\circ, T_\bullet) := \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot c(T_\circ, \delta_\bullet \in T_\bullet) \]

**THM.** For any red triangulation \( T_\circ \), the g-vector fan \( \mathcal{F}^g(T_\circ) \) is the normal fan of

\[ \text{Asso}(T_\circ) = \text{conv} \{ p(T_\circ, T_\bullet) \mid T_\bullet \text{ blue triangulation} \} = \{ x \in \mathbb{R}^{T_\circ} \mid \langle g(T_\circ, \delta_\bullet), x \rangle \leq \omega(\delta_\bullet) \text{ for any blue diagonal} \delta_\bullet \} \].

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
Define $p(T_o, T_\bullet) := \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot c(T_o, \delta_\bullet \in T_\bullet)$

**THM.** For any red triangulation $T_o$, the $g$-vector fan $F^g(T_o)$ is the normal fan of $\text{Asso}(T_o) = \text{conv} \{ p(T_o, T_\bullet) \mid T_\bullet \text{ blue triangulation} \}$

$$= \left\{ x \in \mathbb{R}^{T_o} \mid \langle g(T_o, \delta_\bullet) \mid x \rangle \leq \omega(\delta_\bullet) \text{ for any blue diagonal } \delta_\bullet \right\}.$$  

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
THM. For any red triangulation $T_o$, the $g$-vector fan $\mathcal{F}^g(T_o)$ is the normal fan of

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where

$$p(T_o, T_\bullet) := \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot c(T_o, \delta_\bullet \in T_\bullet) = \sum_{\delta_\circ \in T_o} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{T_o}.$$ 

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* (’17+)

$\implies$ the $\delta_\circ$-coordinate of $p(T_o, T_\bullet)$ does not really depends on $T_o$
**UNIVERSAL ASSOCIAHEDRON**

**THM.** For any red triangulation $T_\circ$, the $g$-vector fan $\mathcal{F}^g(T_\circ)$ is the normal fan of

$$\text{Asso}(T_\circ) = \text{conv} \left\{ p(T_\circ, T_\bullet) \mid T_\bullet \text{ blue triangulation} \right\}$$

where

$$p(T_\circ, T_\bullet) := \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot c(T_\circ, \delta_\bullet \in T_\bullet) = \sum_{\delta_\circ \in T_\circ} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{T_\circ}.$$ 

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)

**THM.** Let $X_\circ$ be the set of all internal red diagonals. Define the universal associahedron $\text{Asso}_{un}(n)$ as the convex hull of the points

$$p_{un}(T_\circ) := \sum_{\delta_\circ \in X_\circ} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{X_\circ}$$

over all blue triangulations. Then for any red triangulation $T_\circ$, the $g$-vector fan $\mathcal{F}^g(T_\circ)$ is the normal fan of the projection $\text{Asso}(T_\circ)$ of the universal associahedron $\text{Asso}_{un}(n)$ on the coordinate plane $\mathbb{R}^{T_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
**THM.** Let $X_\circ$ be the set of all internal red diagonals. Define the universal associahedron $\text{Asso}_{\text{un}}(n)$ as the convex hull of the points

$$p_{\text{un}}(T_\circ) := \sum_{\delta_\circ \in X_\circ} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{X_\circ}$$

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Then for any red triangulation $T_\circ$, the g-vector fan $\mathcal{F}^g(T_\circ)$ is the normal fan of the projection $\text{Asso}(T_\circ)$ of the universal associahedron $\text{Asso}_{\text{un}}(n)$ on the coordinate plane $\mathbb{R}^{T_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)}
**THM.** Let $X_\circ$ be the set of all internal red diagonals.

Define the universal associahedron $\text{Asso}_{\text{un}}(n)$ as the convex hull of the points

$$p_{\text{un}}(T_\circ) := \sum_{\delta_\circ \in X_\circ} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{X_\circ}$$

over all blue triangulations.

Then for any red triangulation $T_\circ$, the g-vector fan $\mathcal{F}^g(T_\circ)$ is the normal fan of the projection $\text{Asso}(T_\circ)$ of the universal associahedron $\text{Asso}_{\text{un}}(n)$ on the coordinate plane $\mathbb{R}^{T_\circ}$.

Hohlweg-P.-Stella, *Polytopal realizations of finite type $g$-vector fans* ('17+)

<table>
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<tr>
<th>$n$</th>
<th>dimension of ambient space</th>
<th>dimension</th>
<th># vertices</th>
<th># facets</th>
<th># vertices / facet</th>
<th># facets / vertex</th>
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<td>1</td>
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<td>$14 \leq \cdot \leq 28$</td>
<td>$3463 \leq \cdot \leq 4244$</td>
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</tbody>
</table>
THM. Let $X_\circ$ be the set of all internal red diagonals. Define the universal associahedron $\text{Asso}_{\text{un}}(n)$ as the convex hull of the points

$$p_{\text{un}}(T_\circ) := \sum_{\delta_\circ \in X_\circ} \left( \sum_{\delta_\bullet \in T_\bullet} \omega(\delta_\bullet) \cdot \varepsilon_\bullet(\delta_\circ, \delta_\bullet \in T_\bullet) \right) e_{\delta_\circ} \in \mathbb{R}^{X_\circ}$$

over all blue triangulations.

Then for any red triangulation $T_\circ$, the $g$-vector fan $\mathcal{F}^g(T_\circ)$ is the normal fan of the projection $\text{Asso}(T_\circ)$ of the universal associahedron $\text{Asso}_{\text{un}}(n)$ on the coordinate plane $\mathbb{R}^{T_\circ}$.

Hohlweg-P.-Stella, Polytopal realizations of finite type $g$-vector fans ('17+)

THM. The origin is the vertex barycenter of the universal associahedron $\text{Asso}_{\text{un}}(n)$.

Hohlweg-P.-Stella, Polytopal realizations of finite type $g$-vector fans ('17+)

CORO. For any red triangulation $T_\circ$, the origin is the vertex barycenter of the $T_\circ$-associahedron $\text{Asso}(T_\circ)$.

Hohlweg-P.-Stella, Polytopal realizations of finite type $g$-vector fans ('17+)
SECTIONS AND PROJECTIONS

Manneville-P., *Geometric realizations of the accordion complex (’17*)
SECTIONS AND PROJECTIONS

THM. For any red triangulation $T_0$, the $g$-vector fan $\mathcal{F}^g(T_0)$ is the normal fan of the projection $\text{Asso}(T_0)$ of the universal associahedron $\text{Asso}_{\text{un}}(n)$ on the coordinate plane $\mathbb{R}^{T_0}$.

What happens if we project on other coordinate planes?
No clue in general, but...

For a red dissection $D_0$, define

$$\text{Asso}(D_0) = \text{projection of } \text{Asso}_{\text{un}}(n) \text{ on the coordinate plane } \mathbb{R}^{D_0}$$

Since normal fan of projections are sections of normal fans,
normal fan of $\text{Asso}(D_0) = \text{section of the normal fan of } \text{Asso}_{\text{un}}(n) \text{ by the plane } \mathbb{R}^{D_0}$

$= \text{subfan of the normal fan of } \text{Asso}_{\text{un}}(n) \text{ induced by the rays in } \mathbb{R}^{D_0}$

$= \text{subfan of the normal fan of } \text{Asso}(T_0) \text{ induced by the rays in } \mathbb{R}^{D_0}$

for a triangulation $T_0$ containing $D_0$.
LEM. For a red dissection $D_\circ$ contained in a red triangulation $T_\circ$, and a blue diagonal $\delta_\bullet$, $g(T_\circ, \delta_\bullet) \in \mathbb{R}^{D_\circ} \iff \delta_\bullet$ never crosses a cell of $D_\circ$ through two non-consecutive edges

$D_\circ$-accordion diagonal = diagonal of the blue solid polygon that crosses an accordion of $D_\circ$

$D_\circ$-accordion dissection = set of non-crossing $D_\circ$-accordion diagonals

$D_\circ$-accordion complex = simplicial complex of $D_\circ$-accordion dissections
THM. For any red dissection $D_\circ$, the projection $\text{Asso}(D_\circ)$ of the universal associahedron $\text{Asso}_{un}(n)$ on the coordinate plane $\mathbb{R}^{D_\circ}$ realizes the $D_\circ$-accordion complex.

Manneville-P., Geometric realizations of the accordion complex ('17+).
If $D_\circ \subseteq D'_\circ$, then

- $\mathcal{F}^g(D_\circ)$ is the section of $\mathcal{F}^g(D'_\circ)$ with the coordinate plane $\langle e_{\delta_\circ} \mid \delta_\circ \in D_\circ \rangle$,
- therefore, $\mathcal{F}^g(D_\circ)$ is also realized by the projection of $\text{Asso}(D_\circ)$ on $\langle e_{\delta_\circ} \mid \delta_\circ \in D_\circ \rangle$. 

PROJECTIONS OF PROJECTIONS
EXTENSIONS TO CLUSTER ALGEBRAS

Fomin-Zelevinsky, *Cluster Algebras I, II, III, IV* ('02–’07)
cluster algebra = commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process:

cluster seed = algebraic data \( \{x_1, \ldots, x_n\} \), combinatorial data \( B \) (matrix or quiver)

cluster mutation = \( (\{x_1, \ldots, x_k, \ldots, x_n\}, B) \xleftarrow{\mu_k} (\{x_1, \ldots, x'_k, \ldots, x_n\}, \mu_k(B)) \)

\[
x_k \cdot x'_k = \prod_{\{i \mid b_{ik} > 0\}} x_i^{b_{ik}} + \prod_{\{i \mid b_{ik} < 0\}} x_i^{-b_{ik}}
\]

\[
(\mu_k(B))_{ij} = \begin{cases} 
-b_{ij} & \text{if } k \in \{i, j\} \\
 b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\
b_{ij} & \text{otherwise}
\end{cases}
\]

cluster complex = simplicial complex w/ vertices = cluster variables & facets = clusters

Fomin-Zelevinsky, Cluster Algebras I, II, III, IV ('02–'07)
CLUSTER MUTATION
$x_4 = \frac{1 + x_2}{x_1}$
\[ x_4 = \frac{1 + x_2}{x_1} \quad \text{and} \quad x_6 = \frac{1 + x_2}{x_3} \]
$x_4 = \frac{1 + x_2}{x_1}$

$x_6 = \frac{1 + x_2}{x_3}$

$x_5 = \frac{x_1 + x_3}{x_2}$
$x_4 = \frac{1 + x_2}{x_1}$

$x_5 = \frac{x_1 + x_3}{x_2}$

$x_6 = \frac{1 + x_2}{x_3}$

$x_9 = \frac{(x_1 + x_3)(1 + x_2)}{x_1 x_2 x_3}$
One constructs a cluster algebra from the triangulations of a polygon:

\[
\begin{align*}
\text{diagonals} & \quad \leftrightarrow \quad \text{cluster variables} \\
\text{triangulations} & \quad \leftrightarrow \quad \text{clusters} \\
\text{flip} & \quad \leftrightarrow \quad \text{mutation}
\end{align*}
\]

\[
\begin{align*}
b & \quad \leftrightarrow \quad b \\
x & \quad \leftrightarrow \quad y \\
d & \quad \leftrightarrow \quad d
\end{align*}
\]

\[
xy = ac + bd
\]
THM. (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.
Fomin-Zelevinsky, *Cluster algebras I: Fundations* ('02)

THM. (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for finite type crystallographic root systems.
Fomin-Zelevinsky, *Cluster algebras II: Finite type classification* ('03)

for a root system \( \Phi \), and an acyclic initial cluster \( X = \{x_1, \ldots, x_n\} \), there is a bijection

\[
y = \frac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \quad \xleftarrow{\theta_X} \quad \Phi_{\geq -1} = \Phi^+ \cup -\Delta
\]

\[
\beta = d_1 \alpha_1 + \cdots + d_n \alpha_n
\]

cluster variables of \( A_\Phi \) \quad \xleftarrow{\theta_X} \quad X\text{-cluster in } \Phi_{\geq -1}

cluster of \( A_\Phi \) \quad \xleftarrow{\theta_X} \quad X\text{-cluster complex in } \Phi_{\geq -1}

see a short introduction to finite Coxeter groups
g- and c-vectors of cluster variables are defined using principal coefficients
universal c-vectors are defined using universal coefficients

**THM.** Σ finite type Dynkin diagram and \( h : \) cluster vars → \( \mathbb{R} \) exchange submodular.
Define the universal Σ-associahedron \( \text{Asso}_{\text{un}}(\Sigma) \) as the convex hull of the points

\[
p_{\text{un}}(\Sigma) := \sum_{x \in \Sigma} h(x) \cdot c_{\text{un}}(x \in \Sigma)
\]

for all seeds \( \Sigma \) in the cluster algebra of type Σ.
Then for any initial seed \( \Sigma_0 \), the g-vetor fan \( \mathcal{F}^g(\Sigma_0) \) is the normal fan of the projection \( \text{Asso}(\Sigma_0) \) of the universal associahedron \( \text{Asso}_{\text{un}}(\Sigma) \) on the coordinate plane \( \mathbb{R}^\Sigma \).

Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
IV. NON-KISSING COMPLEXES AND GENTLE ASSOCIAHEDRA

Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg. ('17+)*
NON-KISSING COMPLEX

Petersen-Pylyavskyy-Speyer, *A non-crossing standard monomial theory* (’10)
Santos-Stump-Welker, *Non-crossing sets and the Grassmann-assoc.* (’17)
McConville, *Lattice structures of grid Tamari orders* (’17)
Garver-McConville, *Enumerative properties of grid-associahedra* (’17+)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* (’17+)
**QUIVERS**

A **quiver** is an oriented graph (loops and multiple edges allowed). It is defined as

\[ Q = (Q_0, Q_1, s, t) \]

where
- \( Q_0 \) are vertices
- \( Q_1 \) are edges
- \( s, t : Q_1 \to Q_0 \) are source and target maps
quiver = oriented graph
(loops and multiple edges allowed)

\( Q = (Q_0, Q_1, s, t) \)
\( Q_0 = \) vertices
\( Q_1 = \) edges
\( s, t : Q_1 \to Q_0 \) source and target maps

**path** = \( \alpha_1 \ldots \alpha_\ell \) with \( \alpha_k \in Q_1 \) and \( t(\alpha_k) = s(\alpha_{k+1}) \)

**path algebra** \( KQ = \langle e_\pi \mid \pi \text{ path of } Q \rangle \) with concatenation product

\[
e_{\alpha_1 \ldots \alpha_\ell} \cdot e_{\beta_1 \ldots \beta_k} = \begin{cases} 
  e_{\alpha_1 \ldots \alpha_\ell \beta_1 \ldots \beta_k} & \text{if } t(\alpha_\ell) = s(\beta_1) \\
  0 & \text{otherwise}
\end{cases}
\]
A **quiver** is an oriented graph (loops and multiple edges allowed).

Let $Q = (Q_0, Q_1, s, t)$ where:

- $Q_0$ are vertices
- $Q_1$ are edges
- $s, t : Q_1 \rightarrow Q_0$ are source and target maps

A **path** is defined as $\alpha_1 \ldots \alpha_\ell$ with $\alpha_k \in Q_1$ and $t(\alpha_k) = s(\alpha_{k+1})$.

The **path algebra** $\mathbb{K}Q = \langle e_\pi \mid \pi \text{ path of } Q \rangle$ has the concatenation product:

$$e_{\alpha_1 \ldots \alpha_\ell} \cdot e_{\beta_1 \ldots \beta_k} = \begin{cases} e_{\alpha_1 \ldots \alpha_\ell \beta_1 \ldots \beta_k} & \text{if } t(\alpha_\ell) = s(\beta_1) \\ 0 & \text{otherwise} \end{cases}$$
**QUIVERS**

quiver = oriented graph  
(loops and multiple edges allowed)

\[ Q = (Q_0, Q_1, s, t) \]

\( Q_0 = \) vertices
\( Q_1 = \) edges
\( s, t : Q_1 \to Q_0 \) source and target maps

---

**Diagram:**

```
1 ← 2
  \downarrow
3 ← 4 ← 6
  \downarrow
5 ← 4
```

---

**Path** = \( \alpha_1 \ldots \alpha_\ell \) with \( \alpha_k \in Q_1 \) and \( t(\alpha_k) = s(\alpha_{k+1}) \)

**Path algebra** \( \mathbb{K}Q = \langle e_\pi \mid \pi \text{ path of } Q \rangle \) with concatenation product

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e_{\alpha_1 \ldots \alpha_\ell \beta_1 \ldots \beta_k} & \text{if } t(\alpha_\ell) = s(\beta_1) \\
0 & \text{otherwise} \end{cases}
\]

**Bound quiver** = quiver with relations
\( \tilde{Q} = (Q, I) \) where \( I \) is an admissible ideal of \( \mathbb{K}Q \).

Complicated way to say that we forbid certain paths
quiver = oriented graph
(loops and multiple edges allowed)

\[ Q = (Q_0, Q_1, s, t) \]

\( Q_0 \) = vertices
\( Q_1 \) = edges
\( s, t : Q_1 \to Q_0 \) source and target maps

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bound quiver = quiver with relations
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Complicated way to say that we forbid certain paths
**QUIVERS**

**quiver** = oriented graph
(loops and multiple edges allowed)

\[ Q = (Q_0, Q_1, s, t) \]

\( Q_0 = \text{vertices} \)
\( Q_1 = \text{edges} \)
\( s, t : Q_1 \to Q_0 \) source and target maps

**path** = \( \alpha_1 \ldots \alpha_\ell \) with \( \alpha_k \in Q_1 \) and \( t(\alpha_k) = s(\alpha_{k+1}) \)

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\]

**bound quiver** = quiver with relations
\( \bar{Q} = (Q, I) \) where \( I \) is an admissible ideal of \( \mathbb{K}Q \).

Complicated way to say that we forbid certain paths
bound quiver $\bar{Q} = (Q, I)$

gentle quiver =
• forbidden paths all of length 2
• locally at each vertex, subgraph of
bound quiver $\bar{Q} = (Q, I)$

gentle quiver =
- forbidden paths all of length 2
- locally at each vertex, subgraph of

blossoming quiver $\bar{Q}^\bullet = \text{add blossoms to complete each vertex to}$
string $\sigma = \alpha_1^{\varepsilon_1} \ldots \alpha_\ell^{\varepsilon_\ell}$

with $\alpha_k \in Q_1$, $\varepsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$
string $\sigma = \alpha_1^{\varepsilon_1} \ldots \alpha_{\ell}^{\varepsilon_{\ell}}$

with $\alpha_k \in Q_1$, 
$\varepsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

substrings of $\sigma = \{ \alpha_i^{\varepsilon_i} \ldots \alpha_j^{\varepsilon_j} \mid 1 \leq i \leq j - 1 \leq k \}$

bottom substring of $\sigma = \text{substring } \rho \text{ of } \sigma \text{ such that } \sigma \text{ either ends }$

or has an outgoing arrow at each endpoint of $\rho$

$\Sigma_{\text{bot}}(\sigma) = \{ \text{bottom substrings of } \sigma \}$

top substring of $\sigma = \text{substring } \rho \text{ of } \sigma \text{ such that } \sigma \text{ either ends }$

or has an incoming arrow at each endpoint of $\rho$

$\Sigma_{\text{top}}(\sigma) = \{ \text{top substrings of } \sigma \}$
string $\sigma = \alpha_1^{\epsilon_1} \ldots \alpha_\ell^{\epsilon_\ell}$

with $\alpha_k \in Q_1$, $\epsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\epsilon_k}) = s(\alpha_k^{\epsilon_k+1})$

walk $\omega =$ maximal string in $Q^\star$

from blossoms to blossoms
string $\sigma = \alpha_1^{\varepsilon_1} \ldots \alpha_\ell^{\varepsilon_\ell}$

with $\alpha_k \in Q_1$, $\varepsilon_k \in \{-1, 1\}$

and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

walk $\omega = \text{maximal string in } Q^*$

from blossoms to blossoms
string $\sigma = \alpha_1^{\varepsilon_1} \cdots \alpha_\ell^{\varepsilon_\ell}$
with $\alpha_k \in Q_1$,
$\varepsilon_k \in \{-1, 1\}$
and $t(\alpha_k^{\varepsilon_k}) = s(\alpha_{k+1}^{\varepsilon_{k+1}})$

walk $\omega = \text{maximal string in } Q^*$
from blossoms to blossoms
walk $\omega = \text{maximal string in } Q^*$
from blossoms to blossoms

$\omega \text{ kisses } \omega' \text{ if } \Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \emptyset$
NON-KISSING COMPLEX

walk $\omega = \text{maximal string in } Q^*$
from blossoms to blossoms

$\omega \text{ kisses } \omega'$ if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \emptyset$
walk $\omega = \text{maximal string in } Q^*$
from blossoms to blossoms

$\omega \text{ kisses } \omega'$ if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \emptyset$
walk $\omega = \text{maximal string in } Q^*$
from blossoms to blossoms

$\omega \text{ kisses } \omega' \text{ if } \Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \emptyset$
**NON-KISSING COMPLEX**

Walk $\omega = \text{maximal string in } Q^*$ from blossoms to blossoms

$\omega \text{ kisses } \omega'$ if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \emptyset$

[reduced] non-kissing complex $\mathcal{K}_{\text{nk}}(\bar{Q}) = \text{simplicial complex with}$
- vertices = [bended] walks of $\bar{Q}$ (that are not self-kissing)
- faces = collections of pairwise non-kissing [bended] walks of $\bar{Q}$
REDUCED NON-KISSING COMPLEX
[reduced] simplicial associahedron = simplicial complex with
- vertices = [internal] diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon
[reduced] simplicial associahedron = simplicial complex with
- vertices = [internal] diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon

diagonal \quad \longleftrightarrow \quad \text{walk}
[reduced] simplicial associahedron = simplicial complex with
- vertices = [internal] diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon
[reduced] simplicial associahedron = simplicial complex with
- vertices = [internal] diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon

diagonal \(\leftrightarrow\) walk
crossing \(\leftrightarrow\) kissing
dissection \(\leftrightarrow\) non-kissing face
**SIMPLICIAL ASSOCIAHEDRA ARE NON-KISSING COMPLEXES**

[reduced] **simplicial associahedron** = simplicial complex with
- vertices = [internal] diagonals of an \((n + 3)\)-gon
- faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon

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McConville, *Lattice structures of grid Tamari orders* ('17)
[reduced] simplicial associahedron = simplicial complex with
• vertices = [internal] diagonals of an \((n + 3)\)-gon
• faces = collections of pairwise non-crossing [internal] diagonals of the \((n + 3)\)-gon
TWO FAMILIES OF NON-KISSING COMPLEXES

dissection

subset of $\mathbb{Z}^2$
TWO FAMILIES OF NON-KISSING COMPLEXES

dissection

subset of $\mathbb{Z}^2$
TWO FAMILIES OF NON-KISSING COMPLEXES

dissection

subset of $\mathbb{Z}^2$

dissection quiver

grid quiver
TWO FAMILIES OF NON-KISSING COMPLEXES

accordion

2457 subset of \([n + m]\)

walk

walk
TWO FAMILIES OF NON-KISSING COMPLEXES

crossing accordions

crossing subsets of $[n + m]$

kissing walks

kissing walks
TWO FAMILIES OF NON-KISSING COMPLEXES

Accordion complex

Grid Tamari complex

Baryshnikov, On Stokes sets (’01)
Chapoton, Stokes posets and serpent nests (’16)
Garver-McConville, Oriented flip graphs and non-crossing tree partitions (’17)

Petersen-Pylyavskyy-Speyer, A non-crossing standard monomial theory (’10)
Santos-Stump-Welker, Non-crossing sets and the Grassmann-assoc. (’17)
McConville, Lattice structures of grid Tamari orders (’17)
Garver-McConville, Enumerative properties of grid-associahedra (’17+)
$F$ face of $\mathcal{K}_{nk}(\bar{Q})$

$\alpha \in Q_1$
\( F \) face of \( \mathcal{K}_{nk}(\bar{Q}) \)
\( \alpha \in Q_1 \)

\[ F_{\alpha} = \{ \omega \in F \mid \alpha \in \omega \} \]

\( \lambda \prec_{\alpha} \omega \) countercurrent order at \( \alpha \)
DISTINGUISHED WALKS, ARROWS AND STRINGS

\[ F \text{ face of } \mathcal{K}_{nk}(\bar{Q}) \]
\[ \alpha \in Q_1 \]
\[ F_\alpha = \{ \omega \in F \mid \alpha \in \omega \} \]
\[ \lambda \prec_\alpha \omega \text{ countercurrent order at } \alpha \]
\[ \text{dw}(\alpha, F) = \max_{\prec_\alpha} F_\alpha \]
\[ \text{da}(\omega, F) = \{ \alpha \in Q_1 \mid \omega = \text{dw}(\alpha, F) \} \]
**PROP.** For any facet $F \in K_{nk}(\bar{Q})$, 
- each bended walk of $F$ contains 2 distinguished arrows in $F$ pointing opposite,
- each straight walk of $F$ contains 1 distinguished arrows in $F$ pointing as the walk.
**PROP.** For any facet $F \in \mathcal{K}_{nk}(\bar{Q})$,
- each bended walk of $F$ contains 2 distinguished arrows in $F$ pointing opposite,
- each straight walk of $F$ contains 1 distinguished arrows in $F$ pointing as the walk.

**CORO.** $\mathcal{K}_{nk}(\bar{Q})$ is pure of dimension $|Q_0|$.
$F$ facet of $\mathcal{K}_{nk}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)
$F$ facet of $\mathcal{K}_{nk}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to “flip”
$F$ facet of $\mathcal{K}_{nk}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to “flip”

$\{\alpha, \beta\} = da(\omega, F)$
Let $F$ be a facet of $\mathcal{K}_{nk}(\bar{Q})$ (i.e. maximal collection of pairwise non-kissing walks). For $\omega \in F$ we want to "flip"

\[ \{\alpha, \beta\} = da(\omega, F) \]

such that $\alpha' \alpha \in I$ and $\beta' \beta \in I$.
**FLIPS**

$F$ facet of $\mathcal{K}_{nk}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to “flip”

$\{\alpha, \beta\} = \text{da}(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = \text{dw}(\alpha, F)$ and $\nu = \text{dw}(\beta, F)$
$F$ facet of $\mathcal{K}_{nk}(\bar{Q})$ (ie. maximal collection of pairwise non-kissing walks)

$\omega \in F$ we want to “flip”

$\{\alpha, \beta\} = da(\omega, F)$

$\alpha', \beta' \in Q_1$ such that $\alpha'\alpha \in I$ and $\beta'\beta \in I$

$\mu = dw(\alpha, F)$ and $\nu = dw(\beta, F)$

$\omega' = \mu[\cdot, v] \sigma \nu[w, \cdot]$
PROP. $\omega'$ kisses $\omega$ but no other walk of $F$. Moreover, $\omega'$ is the only such walk.
GENTLE ASSOCIAHEDRA

Manneville-P., *Geometric realizations of the accordion complex* ('17+)
Hohlweg-P.-Stella, *Polytopal realizations of finite type g-vector fans* ('17+)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* ('17+)
**G-VECTORS & C-VECTORS**

Multiplicity vector $m_V$ of multiset $V = \{v_1, \ldots, v_m\}$ of $Q_0 = \sum_{i \in [m]} e_{v_i} \in \mathbb{R}^{Q_0}$

g-vector $g(\omega)$ of a walk $\omega = m_{\text{peaks}}(\omega) - m_{\text{deeps}}(\omega)$

c-vector $c(\omega \in F)$ of a walk $\omega$ in a non-kissing facet $F = \varepsilon(\omega, F) m_{ds}(\omega, F)$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 \\
2 & 0 & 0 & 0 & 0 & -1 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & -1 & 0 & 0 \\
5 & 0 & 0 & 1 & 0 & 1 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 \\
2 & 0 & 0 & 1 & 0 & -1 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 1 & -1 & 0 & 0 \\
5 & 0 & 0 & 1 & 0 & 0 & 0 \\
6 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

$g_F$, $c_F$
multiplicity vector $m_V$ of multiset $V = \{v_1, \ldots, v_m\}$ of $Q_0 = \sum_{i \in [m]} e_{v_i} \in \mathbb{R}^{Q_0}$
g-vector $g(\omega)$ of a walk $\omega = m_{\text{peaks}(\omega)} - m_{\text{deeps}(\omega)}$
c-vector $c(\omega \in F)$ of a walk $\omega$ in a non-kissing facet $F = \varepsilon(\omega, F) m_{\text{ds}(\omega, F)}$

PROP. For any non-kissing facet $F$, the sets of vectors 

$$g(F) := \{g(\omega) \mid \omega \in F\} \quad \text{and} \quad c(F) := \{c(\omega \in F) \mid \omega \in F\}$$

form dual bases.

Palu-P.-Plamondon, Non-kissing complexes and $\tau$-tilting for gentle algebras (’17+)
THM. For any gentle quiver $\overline{Q}$, the collection of cones $\mathcal{F}^{g}(\overline{Q}) := \left\{ \mathbb{R}_{\geq 0}g(F) \mid F \in \mathcal{C}_{nk}(\overline{Q}) \right\}$ forms a compl. simpl. fan, called \textit{g-vector fan} of $\overline{Q}$.
G-VECTOR FAN
kissing number $\kappa(\omega, \omega') = $ number of times $\omega$ kisses $\omega'$

kissing number $\text{kn}(\omega) = \sum_{\omega'} \kappa(\omega, \omega') + \kappa(\omega', \omega)$

**THM.** For a gentle quiver $\tilde{Q}$ with finite non-kissing complex $C_{nk}(\tilde{Q})$, the two sets of $\mathbb{R}^{Q_0}$ given by

(i) the convex hull of the points

$$p(F') := \sum_{\omega \in F} \text{kn}(\omega) \ c(\omega \in F),$$

for all non-kissing facets $F \in C_{nk}(\tilde{Q})$,

(ii) the intersection of the halfspaces

$$H^\geq(\omega) := \{ x \in \mathbb{R}^{Q_0} \mid \langle g(\omega) \mid x \rangle \leq \text{kn}(\omega) \}.$$

for all walks $\omega$ of $\tilde{Q}$, define the same polytope, whose normal fan is the $g$-vector fan $\mathcal{F}_g$. We call it the $\tilde{Q}$-associahedron and denote it by Asso.

Palu-P.-Plamondon, *Non-kissing complexes and $\tau$-tilting for gentle algebras* ('17+).
NON-KISSING ASSOCIAHEDRON
NON-KISSING ASSOCIAHEDRON VS ZONOTOPES
NON-KISSING LATTICE

McConville, *Lattice structures of grid Tamari orders* (’17)
Palu-P.-Plamondon, *Non-kissing complexes and τ-tilting for gentle alg.* (’17+)
THM. For a gentle quiver $\bar{Q}$ with finite non-kissing complex $C_{nk}(\bar{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.
BICLOSED SETS OF SEGMENTS

σ, τ oriented strings
concatenation \( σ \circ τ = \{ σ\alpha\tau \mid \alpha \in Q_1 \text{ and } σ\alpha\tau \text{ string of } \overline{Q} \} \)
closure \( S^{cl} = \bigcup_{\ell \in \mathbb{N}, \sigma_1, \ldots, \sigma_\ell \in S} \sigma_1 \circ \cdots \circ \sigma_\ell = \text{all strings obtained by concatenation of some strings of } S \)
closed \( \iff S^{cl} = S \) coclosed \( \iff S^{cl} = \overline{S} \) biclosed = closed and coclosed

\[ \text{THM. For any gentle quiver } \overline{Q} \text{ such that } \mathcal{K}_{nk}(\overline{Q}) \text{ is finite, the inclusion poset on biclosed sets of strings of } \overline{Q} \text{ is a congruence-uniform lattice.} \]

McConville, Lattice structures of grid Tamari orders (‘17)
Garver-McConville, Oriented flip graphs and non-crossing tree partitions (‘17+)
Palu-P.-Plamondon, Non-kissing complexes and τ-tilting for gentle algebras (‘17+)
Surjection from biclosed sets of strings to non-kissing facets

\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

\( S \text{ biclosed, } \alpha \in Q_1 \)
\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

\[ \alpha \in S \]
\[ \alpha \notin S \]

\[ \alpha \in S \]
\[ \alpha \notin S \]

McConville, *Lattice structures of grid Tamari orders* ('17)
NON-KISSING INSERTION

Surjection from biclosed sets of strings to non-kissing facets

\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

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McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

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- \( \alpha \notin S \)

McConville, *Lattice structures of grid Tamari orders* ('17)
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\( \omega(\alpha, S) = \) walk constructed with the local rules:

\[
\begin{align*}
\alpha & \in S \\
\alpha & \notin S
\end{align*}
\]

McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

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\alpha & \in S \\
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McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

$S$ biclosed, $\alpha \in \mathcal{Q}_1$

$\omega(\alpha, S) =$ walk constructed with the local rules:

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- $\alpha \notin S$

McConville, *Lattice structures of grid Tamari orders* ('17)
**NON-KISSING INSERTION**

Surjection from biclosed sets of strings to non-kissing facets

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McConville, *Lattice structures of grid Tamari orders* ('17)
Surjection from biclosed sets of strings to non-kissing facets

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$\omega(\alpha, S) =$ walk constructed with the local rules:

$\alpha \in S$

$\alpha \notin S$

$\in S$  $\alpha$

McConville, Lattice structures of grid Tamari orders ('17)
Surjection from biclosed sets of strings to non-kissing facets

\[ S \text{ biclosed, } \alpha \in Q_1 \]
\[ \omega(\alpha, S) = \text{walk constructed with the local rules:} \]

- \( \alpha \in S \)
- \( \alpha \notin S \)

McConville, *Lattice structures of grid Tamari orders* (’17)
Surjection from biclosed sets of strings to non-kissing facets

PROP. $\eta(S) := \{ \omega(\alpha, S) \mid \alpha \in Q_1 \}$ is a non-kissing facet.

McConville, *Lattice structures of grid Tamari orders* ('17)
inversion set of 2751346
inversion set of 2751346
inversion set of 2751346
Surjection from biclosed sets of strings to non-kissing facets

**PROP.** \( \eta(S) := \{ \omega(\alpha, S) \mid \alpha \in Q_1 \} \) is a non-kissing facet.

**THM.** The map \( \eta \) defines a lattice morphism from biclosed sets to non-kissing facets.

McConville, *Lattice structures of grid Tamari orders* ('17)
NON-KISSING LATTICE
**THM.** For a gentle quiver $\widetilde{Q}$ with finite non-kissing complex $\mathcal{C}_{nk}(\widetilde{Q})$, the non-kissing flip graph is the Hasse diagram of a congruence-uniform lattice.

Palu-P.-Plamondon, *Non-kissing complexes and $\tau$-tilting for gentle algebras* ('17+)

Much more nice combinatorics:

- Join-irreducible elements of $\mathcal{L}_{nk}(\widetilde{Q})$ are in bijection with distinguishable strings
- Canonical join complex of $\mathcal{L}_{nk}(\widetilde{Q})$ is a generalization of non-crossing partitions
THANKS
FINITE COXETER GROUPS

Humphreys, *Reflection groups and Coxeter groups* ('90)
Bjorner-Brenti, *Combinatorics of Coxeter groups* ('05)
\[ W = \text{finite Coxeter group} \]
$W =$ finite Coxeter group
Coxeter fan
\[ W = \text{finite Coxeter group} \]

Coxeter fan

fundamental chamber
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber

$S = \text{simple reflections}$
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber

$S = \text{simple reflections}$

$\Delta = \{\alpha_s \mid s \in S\} = \text{simple roots}$
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber

$S = \text{simple reflections}$

$\Delta = \{\alpha_s \mid s \in S\} = \text{simple roots}$

$\Phi = W(\Delta) = \text{root system}$
\[ W = \text{finite Coxeter group} \]
\[ \text{Coxeter fan} \]
\[ \text{fundamental chamber} \]
\[ S = \text{simple reflections} \]
\[ \Delta = \{ \alpha_s \mid s \in S \} = \text{simple roots} \]
\[ \Phi = W(\Delta) = \text{root system} \]
\[ \Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \text{positive roots} \]
$W =$ finite Coxeter group

Coxeter fan

fundamental chamber

$S =$ simple reflections

$\Delta = \{ \alpha_s \ | \ s \in S \} =$ simple roots

$\Phi = W(\Delta) =$ root system

$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] =$ positive roots

$\nabla = \{ \omega_s \ | \ s \in S \} =$ fundamental weights
FINITE COXETER GROUPS

\[ W = \text{finite Coxeter group} \]
\[ \text{Coxeter fan} \]
\[ \text{fundamental chamber} \]
\[ S = \text{simple reflections} \]
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$W =$ finite Coxeter group
Coxeter fan
fundamental chamber
$S =$ simple reflections
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$\Phi = W(\Delta) =$ root system
$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] =$ positive roots
$\nabla = \{ \omega_s \mid s \in S \} =$ fundamental weights
permutahedron
\( W = \text{finite Coxeter group} \)

Coxeter fan

fundamental chamber

\( S = \text{simple reflections} \)

\( \Delta = \{ \alpha_s \mid s \in S \} = \text{simple roots} \)

\( \Phi = W(\Delta) = \text{root system} \)

\( \Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \text{positive roots} \)

\( \nabla = \{ \omega_s \mid s \in S \} = \text{fundamental weights} \)

permutahedron

weak order = \( u \leq w \iff \exists v \in W, uv = w \) and \( \ell(u) + \ell(v) = \ell(w) \)
EXAMPLES: TYPE $A$ AND $B$ 

TYPE $A_n = \text{symmetric group } \mathfrak{S}_{n+1}$

$S = \{(i, i+1) \mid i \in [n]\}$
$\Delta = \{e_{i+1} - e_i \mid i \in [n]\}$
roots = $\{e_i - e_j \mid i, j \in [n+1]\}$
$\nabla = \left\{ \sum_{j>i} e_j \mid i \in [n] \right\}$

TYPE $B_n = \text{semidirect product } \mathfrak{S}_n \rtimes (\mathbb{Z}_2)^n$

$S = \{(i, i+1) \mid i \in [n-1]\} \cup \{\chi\}$
$\Delta = \{e_{i+1} - e_i \mid i \in [n-1]\} \cup \{e_1\}$
roots = $\{\pm e_i \pm e_j \mid i, j \in [n]\} \cup \{\pm e_i \mid i \in [n]\}$
$\nabla = \left\{ \sum_{j\geq i} e_j \mid i \in [n] \right\}$

back to cluster algebras