TWO APPROACHES TO GEOMETRIES IN CHARACTERISTIC 1:

COMBINATORICS vs HOMOTOPY THEORY

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SUMMARY

• After A. Weil's program of studying zeta–functions of varieties over finite fields an exciting challenge emerged: to discover the geometry over an "absolute point" $\mathrm{Spec}\,\mathbf{F}_1$ **which would allow one to apply Weil's philosophy to the study of Riemann's zeta and its global generalisations.**

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• **In this presentation, I will review several attempts to build an** $\mathbf{F}_1$**–geometry, starting with a generalisation/variation of the basic category of commutative rings and proceeding in Weil–Grothendieck style, with higher structures lurking on the backstage.**

• **In particular, starting with certain constructions by Shai Haran, I will then focus upon the idea developed by A. Connes and C. Consani, according to which** *the natural habitat for* $\mathbf{F}_1$*–geometry is the theory of Segal* Γ*–sets* **as a**

combinatorial version of stable homotopy theory.

PLAN

PART I: SHAI HARAN's MODELS: Sections 1–3

S. Haran *New foundations for geometry.* **Memoirs AMS, vol. 246, 2017.**
S. Haran. *Geometry over* $\mathbf{F}_1$. **arXiv:1709.05831**

PART II: CONNES–CONSANI's MODELS: Sections 4–6

A. Connes, C. Consani. *Absolute algebra and Segal's* Γ*–rings au dessous de* $\overline{\mathrm{Spec}\,(\mathbf{Z})}$*.*
arXiv:1502.05585

1. THE BASIC CATEGORY F

• **Definition.** **We consider two equivalent realisations of this category, $\mathbf{F}$ and $\mathbf{F}_*$:**

Realisation 1. $Ob\,\mathbf{F}$**:= finite sets,** $\mathbf{F}(X,Y)$:= **bijective partial maps, i.e. triples** $\varphi, D(\varphi), I(\varphi)$ **where**

$$D(\varphi)\subseteq X,\ \ I(\varphi)\subseteq Y,\ \ \textbf{bijection}\ \varphi : D(\varphi)\to I(\varphi)$$

Realisation 2. $Ob\,\mathbf{F}_*$**:= pointed finite sets** $X_0 = X\cup\{*_X\}$, $\mathbf{F}_*(X,Y)$:= **maps** $\varphi_0 : X\to Y$, **injective on** X.

Equivalence functor: $X\mapsto X\cup\{*_X\}$, $\varphi\mapsto\varphi_0$ **where** φ_0 **agrees with** φ **on** $D(\varphi)$ **and sends** $X\setminus D(\varphi)$ **to** $*_Y$.

• **Exercise.** **Define compositions of morphisms and show that we get an equivalence functor.**

• **Exercise.** **Generalise this construction to various other subcategories of sets with partial maps as morphisms, e. g. finite words in a finite alphabet and semi–computable (partial recursive) maps.**

• **Involutive self – duality functor. It is defined as antiequivalence $()^t : \mathbf{F} \to \mathbf{F}^{op}$: for $\varphi \in \mathbf{F}(X,Y) : D(\varphi) \to I(\varphi)$, we put**

$$\varphi^t = \varphi^{-1} : D(\varphi^t) := I(\varphi) \to I(\varphi^t) := D(\varphi).$$

• **The category F has no sums or products, but has a symmetric monoidal structure $\oplus$ which is the disjoint sum on objects, commuting with self–duality.**

2. COMMUTATIVE RINGS AND THEIR VERSIONS

• **(i)** $CRing$ **:=** *Category of commutative rings, opposite to the category of affine schemes.*

• **(ii)** $CRig$ **:=** *Category of commutative semirings i. e. "commutative rings without negation".*

More precisely, an *object* **of** $CRig$ **is a family** $(R,+,\cdot,0,1)$**, where** R **is a set,** $+$ **and** $\cdot$ **are two commutative associative operations with units** 0**, resp.** 1 **with distributivity, and such that** $0\cdot x=0$ **for all** $x\in R$**.**

Morphisms **are set–theoretic maps compatible with all these structures.**

• **Examples. Objects (subsets of R) and embedding morphisms:**

$$\{0,1\}\subset[0,1]\subset[0,\infty).$$

Operations:

$$x+y:=\max(x,y),\, x\cdot y:=xy.$$

• **Given** $B \in Ob\,CRig$**, we define** *the category* $\mathbf{F}(B)$ *of finitely generated free* B*–modules with a basis.*

Its objects are finite sets indexing the bases, morphisms $X \to Y$ **are given by matrices whose elements belong to** B **and whose lines (resp. columns) are indexed by** Y **(resp.** X**), with composition** $\circ$ **as matrix multiplication in which basic operations** $(\cdot, +)$ **in** B **take part:**

$$Hom_{\mathbf{F}(B)}(X,Y) := \{b = (b_{y,x}) \,|\, y \in Y, x \in X, b_{y,x} \in B\},$$

$$[(b'_{z,y}) \circ (b_{y,x})]_{z,x} = \sum_y b'_{z,y} \cdot b_{y,x}$$

$\mathbf{F}(B)$ **has a symmetric monoidal structure given by direct sums of matrices.**

• *$C\mathcal{G}R$*:= *Category of commutative generalized rings.*

Start with a functor $A : \mathbf{F} \to Sets$**. For** $X, Y \in \mathbf{F}$**,** $f \in Hom_{Sets}(X, Y)$**, put**

$$A_f := \prod_{y \in Y} A_{f^{-1}(y)}.$$

Then we have natural operations:

$$\textit{multiplication: } \triangleleft : A_Y \times A_f \to A_X,$$

$$\textit{contraction: } \| : A_X \times A_f \to A_Y,$$

and their fibrewise extensions: for an additional $g \in Hom_{Sets}(Y, Z)$**,**

$$\triangleleft : A_g \times A_f \to A_{g \circ f}, \quad \| : A_{g \circ f} \times A_f \to A_g.$$

<u>Definition</u>. **(a) A functor A as above is called** *a generalized ring* **iff it satisfies the axioms of** *associativity*

$$a_h \triangleleft (a_g \triangleleft a_f) = (a_h \triangleleft a_g) \triangleleft a_f$$

whenever both sides make sense, and *existence of unit* $1 \in A_{[1]}$*:* **for each** $a \in A_X$**,**

$$1 \triangleleft a = a = a \triangleleft (1)_{x \in X}.$$

(b) A *homomorphism of generalized rings* **is a natural transformation of functors, compatible with multiplications, contractions, and units.**

We thus get the category $\mathcal{G}R$.

(c) A generalized ring is called *commutative* **if we have identically**

$$a \triangleleft (b \parallel b') = (a \triangleleft b) \parallel b'$$

and moreover, if for any $f : X \to Y, g : Z \to Y$**,** $a \in A_X, b \in A_f, c \in A_g$**, we have in** A_Z

$$(a \parallel b) \triangleleft c = (a \triangleleft f^* c) \parallel g^* b.$$

We thus get the category $C\mathcal{G}R$**.**

- **The** *"field with one element"* **in this context is the initial object of** $C\mathcal{G}R$**:**

$$\mathbf{F}_X := X \coprod \{*_X\}, \quad \mathbf{F}_{Y,X} = \mathbf{F}(X, Y).$$

3. GEOMETRIC CONSTRUCTIONS BASED UPON C$\mathcal{G}$R

I will give now a very brief account of results of geometrizations of the categories introduced above.

• One can define affine scheme, localizations and gluing of spectra of objects of $C\mathcal{G}R$: $C\mathcal{G}R$–schemes.

• One can define an important for arithmetics extension of this category to the category of $C\mathcal{G}R$–Pro–schemes.

• One can define formalism of valuations and construct the "zeta–machine" producing p–factors and ∞–factor of the Riemann zeta:

$$\zeta_p(s) = (1 - p^{-s})^{-1} \quad \zeta_{\mathbf{R}}(s) = 2^{s/2}\Gamma(s/2).$$

4. SEGAL′s Γ – SETS

• **Denote by Γ^{op} the category of pointed finite sets $k_+ := \{0, \dots, k\}$, $k = 0, 1, 2, \dots$, whose morphisms are set–theoretic maps sending 0 to 0.**

NB Objects in Γ^{op} are essentially the same as in $\mathbf{F}_*$ from Sec. 1, but there are more morphisms.

• **A Γ–*set* F is a pointed functor between pointed categories $F : \Gamma^{op} \to Sets_*$ where $Sets_*$ is the category of pointed sets.**

• **Definition. The category $\Gamma Sets_*$ consists of Γ–sets as objects and natural transformations of functors as morphisms.**

• **Example. We will now show that Γ–sets generalize commutative monoids (written additively) with zero $(M, +, 0)$.**

Namely, for such M define a Γ–set $HM : \Gamma^{op} \to Sets_*$ by

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On objects k_+: $HM(k_+) := M^k$.
On morphisms $f : k_+ \to n_+$: $HM(f) : M^k \to M^n$, where $HM(f)$ sends $(m_1, \ldots, m_k)$ to $(l_1, \ldots, l_n)$ if

$$l_i = \sum_{j \in f^{-1}(i)} m_j .$$

• **Exercise. Compare this with the definition of generalised rings in Sec. 2 using F in place of Γ^{op}.**

• **Smash products. Let C be a category with internal homomorphisms $\underline{Hom}_C$.**

A *smash product* **in such a category is a bifunctor $\wedge : C \times C \to C$ such that we have a functorial adjunction formula**

$$Hom_C(X \wedge Y, Z) \cong Hom_C(X, \underline{Hom}_C(Y, Z))$$

• **<u>The smash product of pointed sets.</u> It is defined by the simple formula**

$$(X, *_X) \wedge (Y, *_Y) := \text{the result of collapsing } (X \times \{*_Y\}) \cup (\{*_X\} \times Y) \text{ in } X \times Y.$$

• **<u>The smash product of $\Gamma -$ sets.</u> Here I will restrict myself to the description of smash product of two Γ–sets F_1, F_2 in $\mathbf{F}$: the evaluation of $F_1 \wedge F_2$ on a pointed finite set $Z \in Ob\,\mathbf{F}$ is** *the colimit*

$$(F_1 \wedge F_2)(Z) = \text{colim } F_1(X) \wedge F_2(Y)$$

taken over the family of all morphisms $v:\ X \wedge Y \to Z$.

This means that a (non base) point of $(F_1 \wedge F_2)(Z)$ is represented by the quintuple (X, Y, v, x, y) where X, Y are two finite sets with base points; $v : X \wedge Y \to Z$; $x \in F_1(X)$, $y \in F_2(Y)$ two non base points.

Denote now by S the canonical inclusion functor $\Gamma^{op} \to Sets_*$ considered as a Γ–set.

• <u>Definition</u>. **An S–algebra is a Γ–set $\mathcal{A}$ endowed with an associative multiplication with unit**

$$\mu : \mathcal{A} \wedge \mathcal{A} \to \mathcal{A}, \quad 1 : \mathbf{S} \to \mathcal{A}.$$

• *BASIC INTUITION:*

(A) S is $\mathbf{F}_1$ in our framework.

(B) Various subcategories of S-algebras form a natural habitat for algebra "under $Spec\,\mathbf{Z}$".

(C) The smash product $\wedge$ is an incarnation of the imaginary tensor product $\otimes_{\mathbf{F}_1}$.

• **Key examples.** *(a) From monoids to* $\mathbf{S}$*–algebras.*

Let $(M, \cdot, 1; 0)$ **be a multiplicative monoid with unit and absorbing zero, considered as its base point.**

For $X \in Fin_*$**, put** $\mathbf{S}M(X) := M \wedge X$. **Then the product in** M **endows** $\mathbf{S}M$ **with a structure of S–algebra.**

(b) From semirings to $\mathbf{S}$*–algebras.*

Let $(R, (\cdot, 1); (+, 0))$ **be a set with two structures of multiplicative monoids, resp. with unit and zero, considered as its base point. We assume** $+$ **to be commutative, and** $\cdot$ **left and right distributive wrt multiplication.**

For $X \in Fin_*$**, put** $HR(X) := R^{X \setminus \{*\}}$. **Then the functor** $X \mapsto HR(X)$ **is naturally endowed with a structure of S–algebra.**

(c) From hyperrings A/G to $\mathbf{S}$*–algebras.*

A hyperring **is, roughly speaking, a ring with multivalued addition. Natural examples of hyperrings are sets of adèle classes of fields of algebraic numbers.**

Let A be a commutative ring and $G \subset A^\times$ a subgroup of the group of invertible elements of A.

For $X \in Fin_*$, put $HA/G(X) :=$ the quotient of $HA(X)$ by the following equivalence relation:

$\varphi \sim \psi$ *iff there exists $g \in G$ such that for all non base points $x \in X$ we have $\psi(x) = g\varphi(x)$*.

Then the functor $HA/G : \Gamma^{op} \to Sets_*$ is naturally endowed with a structure of S–algebra, and the quotient map $HA \to HA/G$ is a morphism of S–algebras.

Moreover, the classical hyperring A/G can be reconstructed from this S–algebra as $HA/G(1_+)$.

5. EXAMPLE : THE SEMIRING B

• **$\mathbf{B} := (\{0,1\}, \cdot, +)$ in which $1+1=1$ is the smallest semiring of characteristic one. It already leads to quite nontrivial objects.**

• **FACT: $H\mathbf{B}$ is the functor $\Gamma^{op} \to Sets_*$ which associates to an object X the set of subsets of X containing the base point $*_X$.**

$H\mathbf{B} \wedge H\mathbf{B}$ is already a quite nontrivial combinatorial object.

• **A LYRICAL DIGRESSION: Niels Henrik Abel and "characteristic 1".**

Niels Henrik and his siblings were educated by their father [...] for teaching purposes he had handwritten book on history, geography, the mother tongue and mathematics. In mathematics one found not only multiplication and division tables, but also tables for addition and subtraction; in the first line there stood:

$$1+0=0.$$

(from A. Stubhaug, The life of Niels Henrik Abel)

6. $\mathbf{HZ}$ vs $\mathbf{Z}$ and $\mathbf{HZ} \wedge \mathbf{HZ}$ vs $\mathbf{Z} \otimes_{\mathbf{F_1}} \mathbf{Z}$

• **In N. Durov's paper** *New approach to Arakelov Geometry* **, arXiv:0704.2030, a series of definitions and facts involving geometry over $\mathbf{F}_1$ is suggested. However, the basic defect of his theory is the fact that $\mathbf{Z} \otimes_{F_1} \mathbf{Z}$ turns out to be isomorphic to $\mathbf{Z}$.**

• **To the contrary, Connes and Consani prove that $H\mathbf{Z} \wedge H\mathbf{Z}$ is not isomorphic to $H\mathbf{Z}$. The similar statement in the category of the Eilenberg–MacLane spectra also holds.**

THANK YOU FOR YOUR ATTENTION !