$\mathbf{F}_1$:

THE MATHEMATICAL OBJECT

IN SEARCH OF A DEFINITION

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SUMMARY

• I start with a brief history of several fifferent ideas, programs, and constructions, that recently acquired in minds of several mathematicians the traits of Borgesian

"El jardín de senderos que se bifurcan"

"the garden of diverging paths."

• Their list, starting with Gauss' q–deformed binomial coefficients, includes H. Weyl's commutation relations, Jacque Tits' combinatorics of Chevalley groups, M. Kapranov's and A. Smirnov's introduction of $\mathbf{F}_{1^n}$, J. Borger's interpretation of Grothendieck's λ–calculus as descent data "to $\mathbf{F}_1$", et al.

• For more details and more constructions, see the collection

Absolute Arithmetic and $\mathbf{F}_1$*–Geometry*. **Ed. K. Thas, European Math. Soc., 2016,**

including the survey

L. Le Bruyn. *Absolute geometry and the Habiro topology.*, **arXiv:1304.6532**

1. A BRIEF HISTORY

• 1863: From 1 to q – Gaussian binomial coefficients and q–integers

$$[n]_q := q^{n-1} + q^{n-2} + \cdots + 1 = \frac{q^n - 1}{q - 1}, \; n \geq 1.$$

$$[n]_q! := [n]_q [n-1]_q \ldots [1]_q, \; [0]_q! := 1.$$

$$\binom{n}{j}_q := \frac{[n]_q!}{[j]_q! [n-j]_q!}$$

Two natural habitats of q–integers:

(i) *Quantum mechanics/noncommutative geometry habitat:* **define the q–commutator by**

$$[x, y]_q := xy - q^{-1} yx$$

Natural values of q in quantum physics: $q = e^{2\pi i/h}, |q| = 1$**.**

q**–binomial formula: if** $[x, y]_q = 0$ **(Weyl's commutation relation), then**

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^{n-j} y^j .$$

Limit $q \to 1$ **= classical physics.**

(ii) *Finite geometries habitat:*

$$\operatorname{card} \mathbf{P}^{n-1}(\mathbf{F}_q) = \frac{\operatorname{card} (\mathbf{A}^n(\mathbf{F}_q) \setminus \{0\})}{\operatorname{card} \mathbf{G}_m(\mathbf{F}_q)} = \frac{q^n - 1}{q - 1} = [n]_q,$$

$$\operatorname{card} Gr\,(n, j)(\mathbf{F}_q) = \operatorname{card} \{\mathbf{P}^j(\mathbf{F}_q) \subset \mathbf{P}^n(\mathbf{F}_q)\} = \binom{n}{j}_q .$$

Natural values of q in (finite) geometries: $q = p^k$**,** p **prime,** $k \geq 1$**.**
Limit $q \to 1$ **= our imaginary geometry over** $\mathbf{F}_1$**.**

• 1957: From q to 1 – Jacques Tits' projective space

$\mathbf{P}^{n-1}(\mathbf{F}_1)$:= a finite set P of cardinality n.

$Gr\,(n,j))(\mathbf{F}_1)$ **:= the set of subsets of P of cardinality j.**

If one puts $q = 1$ in the previous formulas, all cardinalities agree!

Tits' program: **make sense of algebraic geometry over** *"a field of characteristic one"*
so that the "projective geometry" above becomes a special case of the geometry of Chevalley groups and their homogeneous spaces.

NB: The first implementation of Tits' program appears only in 2008:
A. Connes, C. Consani, arXiv: 0809.2926

However, as s a field of definition C.–C. need $\mathbf{F}_{1^2}$, not just $\mathbf{F}_1$!

• 1991: $\mathbf{F}_{1^n}$ – Kapranov–Smirnov

$\mathbf{F}_{1^n}$ **"is"** *the monoid* $\{0\} \cup \mu_n$**.**

A vector space over $\mathbf{F}_{1^n}$ **is** *a pointed set* $(V, 0)$ *with an action of* μ_n *free on* $V \setminus \{0\}$*.*
Invertible linear maps = permutations compatible with action.

Example. **If** $q \equiv 1 \bmod n$ **and** μ_n **is embedded in** $\mathbf{F}_q^*$**,** $\mathbf{F}_q$ **becomes a vector space over** $\mathbf{F}_{1^n}$**, and the power residue symbol**

$$\left(\frac{a}{\mathbf{F}_q}\right)_n := a^{\frac{q-1}{n}} \in \mu_n$$

is *the determinant of the multiplication by* a **in** $\mathbf{F}_{1^n}$**–geometry.**

• 1992: $\mathbf{Z} \otimes_{\mathbf{F}_1} \mathbf{Z}$ and zeta : Yu. M.

(i) We need an *"absolute point"*, **say** $Spec\,\mathbf{F}_1$, **in arithmetic geometry: then we could try to imitate Weil's proof of Riemann Conjecture for curves** C **over finite fields using intersection theory on** $C \times C$.

(ii) There might exist *a category of motives* **over** $\mathbf{F}_1$, **partly visible through their zeta–functions. The zetas of non–negative powers of the "Lefschetz (dual Tate) motive"** $\mathbf{L}$ **must be:**

$$Z(\mathbf{L}^{\times n}, s) = \frac{s+n}{2\pi}.$$

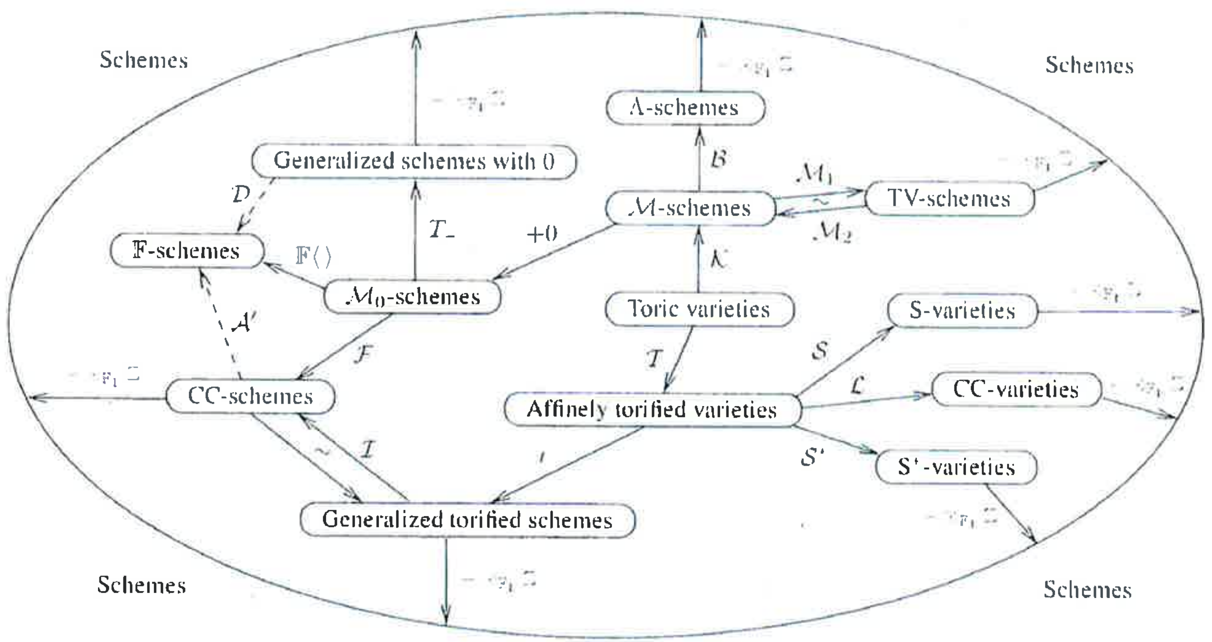
(iii) This provides a conjectural *bridge between* $\mathbf{F}_1$*–geometry* **and geometry of** $Spec\,\mathbf{Z}$ **at the archimedean infinity, that is,** *Arakelov geometry*: **the** Γ**–factor of classical zetas**

$$\Gamma_{\mathbf{C}}(s) := [(2\pi)^{-s}\Gamma(s)]^{-1} = \prod_{n\geq 0} \frac{s+n}{2\pi}$$

(regularized product) looks like $\mathbf{F}_1$*–zeta of inf–dim projective space over* $\mathbf{F}_1$.

- **1999, 2004 and later: Emergence of categories of schemes/$\mathbf{F}_1$:
Soulé, Toën–Vaquié, Deitmar, Haran, Durov, Connes–Consani ...**

• **DEFINITIONS FOUND! (Too many ...)**



LORSCHEID – PEÑA MAP OF F_1–LAND, 2009

. . . but new constructions "in characteristic 1" that do not fit any of the definitions keep appearing.

2. CYCLOTOMY
AND ANALYTIC GEOMETRY OVER $\mathbf{F_1}$

• **Introduction : roots of unity and Morse – Smale diffeomorphisms.**

(i) Let M be a compact smooth manifold, f a diffeomorphism of M. It is called *Morse–Smale*, **if it is structurally stable, and only a finite number of points x are non–wandering, i. e., for any neighborhood U of x, we have $U \cap f^n(U) \neq \emptyset$ infinitely often.**

Assume that all eigenvalues of the action of f on integral cohomology of M are roots of unity and put the question: *when f is isotopic to a Morse–Smale map?*

There is an obstruction to this, lying in the group $SK_1(\mathcal{R})$, where $\mathcal{R}$ is the ring obtained by localizing $\mathbf{Z}[q]$ with respect to $\Phi_0(q) := q$ and all cyclotomic polynomials

$$\Phi_n(q) := \prod_{\eta} (q - \eta)$$

where η runs over all primitive roots of unity of degree $n \geq 1$.

(ii) This ring $\mathcal{R}$ turns out to be a *principal ideal domain.* **The reason for this is that each closed point (a prime ideal of depth two) of the "arithmetical plane" $Spec\,\mathbf{Z}[q]$ is situated on an arithmetic curve $\Phi_n(q) = 0, n \geq 0$, because all finite fields consist of roots of unity and zero.**

Localization *cuts all these curves off,* **and all closed points go with them. Remaining prime ideals are of height one, and they are principal ones.**

(iii) The same effect is achieved by *localizing wrt all primes* $p \in \mathbf{Z}$**, thus getting the principal ideal domain $\mathbf{Q}[q]$. This localization cuts away the closed fibers of the projection $Spec\,\mathbf{Z}[q] \to Spec\,\mathbf{Z}$, and all the closed points with them.**

This suggests that the union of all cyclotomic arithmetic curves $\Phi_n(q) = 0$ can be imagined as the union of closed fibers of the projection $Spec\,\mathbf{Z}[q] \to Spec\,F_1[q]$, and the arithmetic plane itself as *the product of two coordinate axes*, **arithmetic one $Spec\,\mathbf{Z}$ and geometric one, $Spec\,F_1[q]$, over the "absolute point" $Spec\,F_1$.**

Question. **Is there a context in which diffeomorphisms f, acting on integral cohomology of M with eigenvalues roots of unity, could be interpreted as "Frobenius maps in caracteristic 1", and their fixed (or non–wandering) points in a Morse–Smale situation as F_{1^n}–points of an appropriate variety?**

• **<u>Habiro′s analytic functions of many variables</u>.**

<u>Notations</u>. *Rings* **in this section are associative, commutative and unital.** *Ring homomorphisms* **are unital. Letters** $R, R_0, R_1 \dots$ **denote rings,** $q, q_0, q_1 ...$ **are independent commuting variables.**

Let R **be a ring,** $\mathcal{I} = \{I_\alpha\}$ **a family of ideals filtered by inclusion. The projective limit** $\varprojlim_\alpha R/I_\alpha$ **is called** *the completion of* R *with respect to* $\mathcal{I}$ **and denoted** $\widehat{R}_{\mathcal{I}}$**. When** $\mathcal{I}$ **is (cofinal to) the family of powers of one ideal** I**, the respective limit is called the** I**–adic completion.**

We say that R **is** $\mathcal{I}-$ **(resp.** I**–adically)** *separated***, if** $\cap_\alpha I_\alpha = \emptyset$**. Equivalently, the canonical homomorphism** $R \to \widehat{R}_{\mathcal{I}}$ **is injective.**

When q **is considered as a "quantization parameter", our q-deformed versions of integers and factorials will be here**

$$\{N\}_q := q^N - 1,\ \{N\}_q! := \{N\}_q \{N-1\}_q \dots \{1\}_q .$$

Fix an integral domain R_0 of characteristic zero and put $R_n := R_0[q, \dots, q_n]$, with natural embeddings $R_0 \subset R_1 \subset R_2 \subset \dots$

Denote by $I_{n,N} \subset R_n$ the ideal $(\{N\}_{q_1}!, \dots, \{N\}_{q_n}!)$, $N \geq 1$. Clearly, $I_{n,N} \subset I_{n,N+1}$ so that the rings $R_n^{(N)} := R_n/I_{n,N}$, n being fixed, form an inverse system.

• **Definition**. *The ring of Habiro's analytic functions of n variables over R_0 is defined as*

$$\widehat{R}_n := \varprojlim_N R_n^{(N)}.$$

• **Taylor series of analytic functions. Choose a vector of roots of unity $\zeta = (\zeta_1, \dots, \zeta_n)$ such that all ζ_i are in R_0.**

For each integer $M > 0$, there exists $N_0 = N_0(\zeta, M)$ such that for all $N \geq N_0$

$$I_{n,N} \subset (q_1 - \zeta_1, \dots, q_n - \zeta_n)^M$$

In fact, $\{N\}_{q_i}!$ is divisible by any fixed monomial $(q_i-\zeta)^M, \zeta \in \mu$, if N is large enough.

The completion $\varprojlim_M R_n/(q_1-\zeta_1,\dots,q_n-\zeta_n)^M$ is $R[[q_1-\zeta_1,\dots,q_n-\zeta_n]]$.

Therefore we obtain a ring homomorphism "Taylor expansion at the point ζ":

$$T_n(\zeta):\ \widehat{R}_n \to R_0[[q_1-\zeta_1,\dots,q_n-\zeta_n]].$$

• **<u>Theorem</u>.** *If R_0 is an integral domain, p–adically separated for all primes p, then the same is true for $\widehat{R}_n$, and the Habiro–Taylor homomorphism $T_n(\zeta)$ is injective.*

More generally, let $F=\{F_1,\dots,F_n\}\in \mathbf{Z}[q]$ be a family of monic polynomials in $R_0[q]$ whose all roots are roots of unity. Denote by (F) the ideal generated by $F_1(q_1),\dots,F_n(q_n)$ in R_n. In place of the formal series ring above, we can consider the completion

$$\widehat{R}_F := \varprojlim_M R_n/(F)^M$$

and the respective Taylor expansion homomorphism:

$$T_n(F) : \widehat{R}_n \to \widehat{R}_F.$$

• **<u>Theorem</u>.** *If R_0 is an integral domain, p–adically separated for all p, $R[[F]]$ is as well p–adically separated, and the homomorphism $T_n(F)$ is injective.*

• **<u>Differential calculus</u>. Divided powers of partial derivatives with respect to q_k are continuous wrt linear topologies generated by $I_{n,N}$, resp. by all $(q_1 - \zeta_1, \dots, q_n - \zeta_n)^M$. Hence these derivatives make sense in $\widehat{R}_n$, and their values at $(\zeta_1, \dots, \zeta_n)$ are** *the Taylor coefficients of the respective series.*

Thus we can develop for $\widehat{R}_n$ the conventional formalism of tangent and cotangent modules, differential forms etc.

• **<u>Elements of $\widehat{\mathbf{R}}_{\mathbf{n}}$ as functions on roots of unity</u>. Let $R_0' \supset R_0$ be an integral domain flat over R_0 and containing all roots of unity (that is, all cyclotomic polynomials $q^n - 1$ completely split in R_0').**

Denote by μ the set of all roots of unity in R_0'. Choose $\zeta := (\zeta_1, \dots, \zeta_n) \in \mu^n$. Any element of R_n, being a polynomial in $(q_1, \dots, q_n)$, takes a certain value at ζ belonging to R_0'.

If $N \geq N_0(\zeta)$, all elements of $I_{n,N}$ vanish at ζ. Hence any element $f \in \widehat{R}_n$ defines a map $\bar{f} : \mu^n \to R_0'$. This map is R_0–linear and compatible with pointwise addition and multiplication of functions.

Besides assuming that R_0 is p–adically separated for all primes p, impose the following separatedness condition: *for any infinite sequence of pairwise distinct primes $p_1, \dots, p_k, \dots,$ we have*

$$\cap_{m=1}^{\infty} R p_1 \dots p_m = \{0\}.$$

• **Theorem**. *Under these assumptions, the map $f \mapsto \bar{f}$ is injective.*

K. Habiro has also shown that vanishing of $\bar{f}$ on certain sufficiently large subsets of μ suffices to establish the vanishing of f.

More precisely, *Habiro's topology* **on the set μ of all roots of unity is defined as follows. Two roots of unity ξ, η are called** *adjacent*, **if $\xi\eta^{-1}$ is of order p^m, $m \in \mathbf{Z}$, p a prime; or equivalently, if $\xi - \eta$ is not a unit (as an algebraic number). Clearly, the action of $Gal\,(\overline{\mathbf{Q}}/\mathbf{Q})$ preserves adjacency.**

• **Definition**. *A subset $U \subset \mu$ is called open, if for any point $\xi \in U$, all except of finitely many $\eta \in \mu$ adjacent to ξ, belong to U.*

The Galois action is continuous in this topology, in marked contrast to the topology induced from $\mathbf{C}$.

Let now μ' be an infinite set of roots of unity. A point $\xi \in \mu'$ is a limit point of μ', if for any open neighborhood U of ξ we have $\mu' \cap (U \setminus \xi) \neq \emptyset$. In Habiro's topology, this means that μ' contains infinitely many points, adjacent to ξ.

• **Theorem.** *Let $\nu = \nu_1 \times \cdots \times \nu_n \subset \mu^n$ be a set, such that each $\nu_i \subset \mu$ has a limit point. Let $f \in \widehat{R}_n$. If the restriction $\bar{f}|_\nu$ is identical zero, then $f = 0$.*

• **Analogs of Habiro′s functions on the arithmetic axis and analytic continuation.**

The Habiro ring of one variable $\varprojlim_N \mathbf{Z}[q]/(\{N\}_q!)$ "is" the lift to $\mathbf{Z}$ of an imaginary ring $\varprojlim_N F_1[q]/(\{N\}_q!)$.

Along the arithmetical axis, the straightforward analog of the latter exists: this is the topological ring of profinite integers $\widehat{\mathbf{Z}} := \varprojlim_N \mathbf{Z}/(N!)$. Its elements can be uniquely represented by infinite series $\sum_{n=1}^{\infty} c_n n!$ where c_n are integers $0 \leq c_n \leq n$.

An analog of the profinite number $1+\sum_{n=1}^{\infty}(-1)^n n!$ is the remarkable example of Habiro function of one variable

$$1+\sum_{n=1}^{\infty}(-1)^n\{n\}_q! = 1+\sum_{n=1}^{\infty}(1-q)\dots(1-q^n).$$

Considered as a function on roots of unity, it emerged in a work of M. Kontsevich on Feynman integrals (talk at MPIM, 1997). Don Zagier proved that its values, as well as values of its derivatives, are radial limits of the function (resp. its derivatives) holomorphic in the unit circle

$$\frac{1}{2}\sum_{n=1}^{\infty} n\chi(n)q^{(n^2-1)/24},$$

where χ is the quadratic character of conductor 12.

L–FUNCTIONS AND ZETA POLYNOMIALS

• **Notations.**

$U(z) \in \mathbf{R}[z]$ **a polynomial of degree** $e \geq 1$**,** $U(1) \neq 0$**,**

$P(z) := \frac{U(z)}{(1-z)^d}$, $d > e$**,**

$H(x)$ **:= the polynomial of degree** $d-1$ **such that if for** $|z| < 1$ **we have**

$$P(z) := \frac{U(z)}{(1-z)^d} = \sum_{n=0}^{\infty} h_n z^n,$$

then

$H(n) = h_n$ **for all** $n \geq \max\{0, e-d+1\}$**.**

• **<u>Theorem</u>.** *(Popoviciu, Rodriguez–Villegas, et al.) Assume that all roots of U lie on the unit circle.*

Then H satisfies the zeta–type functional equation

$$H(x) = (-1)^{d-1} H(-d+e-x),$$

and vanishes at integer points $x = -1, \dots, -d+e+1$ inside its "critical strip".

Moreover, all the remaining zeroes lie on the vertical line passing through the middle of the critical strip:

$$\mathrm{Re}(x) = \frac{-d+e+1}{2}.$$

• **<u>Examples</u> : Hilbert polynomials H of certain graded commutative rings, in particular anticanonical rings of Fano varieties, and their versions for Calabi–Yau varieties.**

• **<u>Heuristics : relationship between H and P as a "discrete Mellin transform".</u>**

– Classical Mellin transform: $f(z)$ **a Fourier series in upper half–plane, e. g. a cusp form,** $Z_f(s)$ **the associated Dirichlet series:**

$$Z_f(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{i\infty} f(z) \left(\frac{z}{i}\right)^{s-1} d\left(\frac{z}{i}\right).$$

– Integral representation of the passage from $P(z)$ **to** $H(x)$**:**

$$H_f(n) = \frac{1}{2\pi i} \int_\gamma P_f(z) z^{-(n+1)} dz,$$

where γ **is a small contour around zero.**

• **NB This explains also that** s **corresponds to** $-n$**.**

ZETA POLYNOMIALS FROM MODULAR FORMS

• **I will call polynomials H as above zeta polynomials.**

Below I will show how to obtain zeta polynomials from those $PSL(2,\mathbf{Z})$–cusp forms f that are eigenforms for all Hecke operators.

• **Heuristics : These zeta polynomials correspond to Euler factors of the respective L–series "in characteristics one".**

• **Period polynomials. Let f be a cusp form of positive even weight k, $w := k+2$. Put**

$$r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau, \quad r_f^{\pm}(z) := \frac{r_f(z) \pm r_f(-z)}{2}.$$

• **Proposition**. *(J. B. Conrey, D. W. Farmer, Ö. Imamoglu).*
Let f be a cusp Hecke eigenform of weight k with real Fourier coefficients. Then

$$U_f(z) := \frac{r_f^-(z)}{z(z^2-4)(z^2-1/4)(z^2-1)^2} \in \mathbf{R}[z]$$

is a polynomial of degree $e := w - 10$ without real zeros whose complex zeros all lie on the unit circle.

• **<u>Final result</u>**. *Fix an integer* $d > e = w - 10$ *and put*

$$P_f(z) := \frac{U_f(z)}{(1-z)^d} = \sum_{n=0}^{\infty} h_f(n) z^n.$$

Let $H_f(x) \in \mathbf{R}[x]$ *be the polynomial of degree* $d-1$ *such that*

$$h_f(n) = H_f(n)$$

for $|z| < 1$ *and* n *big enough.*

This polynomial satisfies the functional equation

$$H_f(x) = (-1)^{d-1} H(-d+e-x)$$

and vanishes at $x = -1, \ldots, -d+e+1$. *All its remaining zeros lie on the vertical line* $Re\, x = -(d-e-1)/2$.

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THANK YOU FOR YOUR ATTENTION !