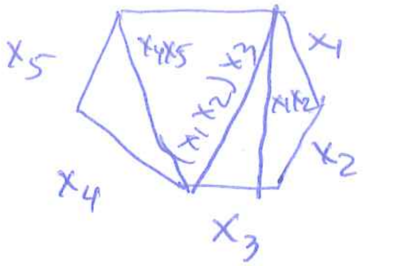
# Associativity and higher Segal structures

w. T. Dyckerhoff.

Brackettings of $x_1 \dots x_n$ $\longleftrightarrow$ Triangul. of $(n+1)$ gon

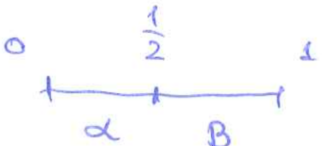
$((x_1x_2)x_3)(x_4x_5)$

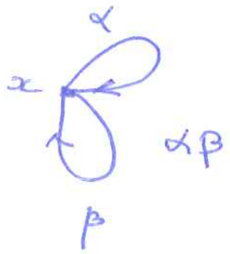


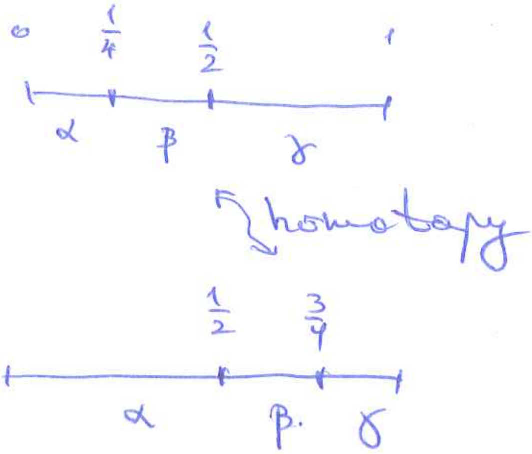

$K_n$ Stasheff polytope (associahedron)

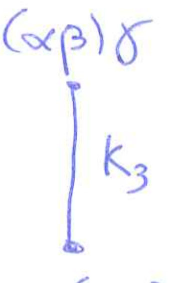
Reason why smth. is associative: always interesting!

Ex. 1. Why $\pi_1(X, x)$ is associative?




$K_n \hookrightarrow \underbrace{\text{Homeo}(I, I)}^{\text{(Torsor over)}}$

$\text{Vert}(K_n) \hookrightarrow$ Thompson group

$K_n$: finite-dim approx of $\infty$-dim group

Ex.2 Hall algebras

$\mathcal{A}$ abelian category "finitistic" — $\mathrm{Hom}(A,B)$, $\mathrm{Ext}^1(A,B)$ finite sets

e.g. {finite ab. groups}, or ab. p-groups

$k=\mathbb{F}_q$ $\quad \mathrm{Rep}_k(\Gamma)$ representations $= k\Gamma$-mod
quiver

$X/_k$ proj. var. $\mathrm{Coh}(X)$

$H(\mathcal{A})$ $\quad \mathbb{C}$-vect. space basis $e_A$ $\quad A \in \mathrm{Ob}\,\mathcal{A}/\mathrm{Iso}$

Mult.
$$e_A \cdot e_B = \sum_{\substack{C \\ \in \\ \mathrm{Ob}\,\mathcal{A}/\mathrm{iso}}} g_{AB}^C \, e_C$$

$g_{AB}^C = \#$ sub-objects $A' \subset C$
$A' \simeq A$ type $A$
$C/A' \simeq B$ cotype $B$

Ex1 $\mathcal{A} = k$-Vect $\quad k=\mathbb{F}_q$

$e_n = e_{k^n} \qquad e_m \cdot e_n = \binom{m+n}{m}_q e_{m+n}$

$\binom{m+n}{m}_q = \# G(m, \mathbb{F}_q^{m+n})$

divided power algebra

$$e_n = \frac{e_1^n}{[n!]_q}$$

<u>Ex. 2 (Ringel)</u> $\mathcal{A} = \mathrm{Rep}_k \Gamma$

say $\Gamma$ Dynkin quiver ADE $\quad \mathfrak{g}_\Gamma$ corr. simple Lie

$$H(\mathrm{Rep}_k \Gamma) = U_q(\mathfrak{n}^+_\Gamma)$$

pos. part. of quantum grp.

$$\mathfrak{g}_\Gamma \supset \mathfrak{n}^+_\Gamma \text{ nilpotent}$$

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \subset \mathfrak{sl}_{n+1}$$

$$\bullet \longrightarrow \overset{i}{\bullet} \longrightarrow \qquad \Gamma = A_n$$

$i \in \mathrm{Vert}(\Gamma) \rightsquigarrow k_i$ simple module

$$e_i = e_{k_i} \in H(\mathcal{A})$$

$[e_i, e_j] = 0$ if $i, j$ not connected

$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ & a cubic rel. if connected

q-analog of $\underbrace{\phantom{x}}_{\text{Serre rel.}}$ $[e_i[e_i, e_j]] = 0$ in $\mathfrak{g}$

- - - -

<u>Why associative?</u> Short answer: b/c

$(e_A e_B) e_C$ and $e_A(e_B e_C)$ both $= \sum g^{\mathcal{D}}_{ABC}\, e_{\mathcal{D}}$

Filtrations "emerge" in two different ways

$g^{\mathcal{D}}_{ABC} = \#$ filtrations $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}$

$$\mathcal{D}_1 \simeq A, \quad \mathcal{D}_2/\mathcal{D}_1 \simeq B, \quad \mathcal{D}/\mathcal{D}_2 \simeq C$$

Want better understanding in order to

— Take $k = \mathbb{C}$, not $\mathbb{F}_q$, and

for $g_{AB}^C$

Use $\chi_{top}$ (space of subobjects), not # of subobj

Use not $\bigoplus \mathbb{C} \cdot e_A = \mathrm{Fun}_{\mathbb{C}}(\mathrm{Ob}\,\mathcal{A}/\mathrm{iso})$

funct. with finite sup

but $H_c^\bullet$ (moduli space / stack of objects)

(Cohomological Hall alg.)

— Use more general categories $\mathcal{A}$ (not abelian)

sub-object $\sim$ ?   triangulated

<u>Groupoid interpretation</u>

$\mathcal{A}^\sim = (\mathcal{A}$, isomorphisms only) groupoid

Orbifold $\overset{\text{def}}{=}$ groupoid $\mathcal{G}$ with all $|\mathrm{Aut}(x)| < \infty$

$$\mathcal{F}_0(\mathcal{G}) = \text{fin. supp. functions } f : \mathrm{Ob}\,\mathcal{G}/_{\mathrm{iso}} \to \mathbb{C}$$

$$\int_{\mathcal{G}} f = \sum_{x \in \mathrm{Ob}\,\mathcal{G}/_{\mathrm{iso}}} \frac{f(x)}{|\mathrm{Aut}(x)|}$$

$\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ functor of groupoids

$$\mathcal{F}_0(\mathcal{G}) \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} \mathcal{F}_0(\mathcal{H})$$

$\varphi^*$: pullback

$\mathcal{G}/y$ = group. of pairs $x \in \mathrm{Ob}\,\mathcal{G}$, $\varphi(x) \xrightarrow{\sim} y$

defined when $\varphi$ <u>proper</u> :

$(\mathcal{G}/y)/\mathrm{iso}$ finite

$\{x_1 \in \mathcal{G}_1, x_2 \in \mathcal{G}_2, p_1(x_1) \xrightarrow{\sim} p_2(x_2)\}$

$$(\varphi_* f)(y) = \int_{\left\{ \begin{smallmatrix} x \in \mathcal{G} \\ \varphi(x) \xrightarrow{\sim} y \end{smallmatrix} \right\} \in \mathcal{G}/y} f(x)$$

Usual properties e.g. base change

$$\begin{tikzcd} \mathcal{G}_1 \overset{h}{\underset{\mathcal{H}}{\times}} \mathcal{G}_2 = \mathcal{G}_{12} \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\ \mathcal{G}_1 \arrow[dr, "p_1"'] & \mathcal{G}_2 \arrow[d, "p_2"] \\ & \mathcal{H} \end{tikzcd}$$

$\mathcal{G}_{12}$ – "homotopy" fiber product

$$q_{2*} q_1^* = p_2^* p_{1*}$$

SES($\mathcal{A}$) = groupoid of

$$S = \{ 0 \to A \to B \to C \to 0 \}$$

$$\begin{tikzcd} & 1 \arrow[dr, "\partial_0"] & \\ 0 \arrow[ur, "\partial_2"] \arrow[rr, "\partial_1"'] & & 2 \end{tikzcd}$$

$$A = \partial_2 S, \quad B = \partial_1 S, \quad C = \partial_0 S$$

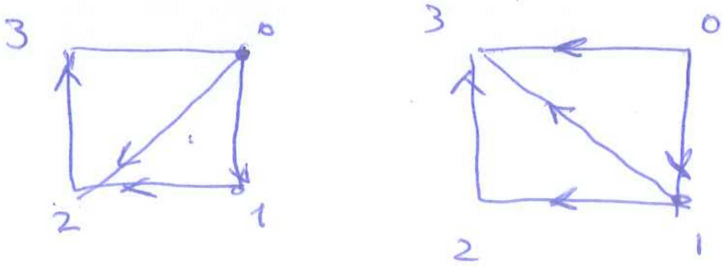
$$\mathrm{SES}(\mathcal{A}) \xrightarrow{\partial_0, \partial_1, \partial_2} \mathcal{A}^\sim$$

$$\begin{tikzcd} & \mathrm{SES}(\mathcal{A}) \arrow[dl, "{(\partial_2, \partial_0)}"'] \arrow[dr, "\partial_1"] & \\ \mathcal{A}^\sim \times \mathcal{A}^\sim & & \mathcal{A}^\sim \end{tikzcd}$$

$$\begin{tikzcd} H(\mathcal{A}) \otimes H(\mathcal{A}) \arrow[rr, "m"] \arrow[d, equal] & & H(\mathcal{A}) \arrow[d, equal] \\ \mathcal{F}_0(\mathcal{A}^\sim \times \mathcal{A}^\sim) \arrow[r, "{(\partial_2, \partial_0)^*}"] & \mathcal{F}_0(\mathrm{SES}(\mathcal{A})) \arrow[r, "\partial_{1*}"] & \mathcal{F}_0(\mathcal{A}^\sim) \end{tikzcd}$$

Underlying associativity of $H(\mathcal{A})$ is

Flip property.



$$SES(\mathcal{A}) \underset{\partial_0 \;\; \partial_2}{\overset{h}{\times}} SES(\mathcal{A}) \simeq SES(\mathcal{A}) \underset{\partial_0 \;\; \partial_1}{\overset{h}{\times}} SES(\mathcal{A})$$

$\mathcal{A}^{\sim}$ $\qquad$ $\mathcal{A}^{\sim}$

(= groupoid of 2-term filtrations)

Which holds for any $\mathcal{A}$, finitistic or not.

## Language of simplicial objects

Simplicial set $X$: sets $X_n$, $n \geq 0$ (of $n$-simplices)

$\partial_i : X_n \to X_{n-1}$ $i = 0, \dots, n$ faces

$s_i : X_n \to X_{n+1}$ degeneracies

relations $\partial_i \partial_j = \partial_{j-1} \partial_i$ $\quad i > j$ ...

contravar.

= functors $\Delta \longrightarrow Set$

$\shortparallel$

finite ordinals $[n] = \{0, 1 \dots n\}$
and monotone maps

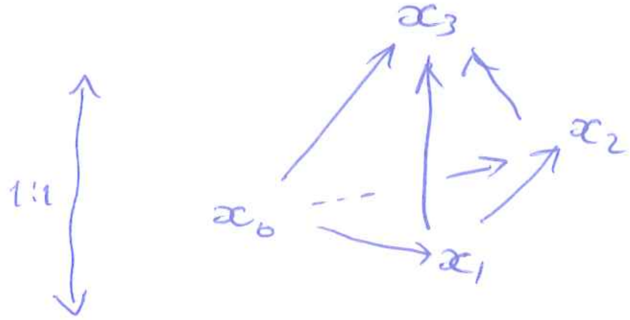
$\partial_i \leftarrow\rightarrow$ $\delta_i : [n-1] \longrightarrow [n]$

missing $i$

S.sets
form Category $\Delta^\circ set$

<u>Ex.</u> $\mathcal{C}$ category $\rightsquigarrow$ Nerve $N.\mathcal{C}$ simpl. set

$N_n\mathcal{C} = \{$commutative $n$-simplices in $\mathcal{C}\}$




$\forall$ 2 face comm

$x_0 \to x_1 \to \cdots \to x_n$ chains of composables

Simplicial objects in any category $\mathcal{E}$: similarly

$X_n \in \mathcal{E}$ $\quad \partial_i, s_i$ morphisms in $\mathcal{E}$

$\Delta^\circ\mathcal{E}$ : their cat. $= \mathrm{Fun}^\circ(\Delta, \mathcal{E})$

Simplicial top. spaces
vect. spaces
groups
- - -

Simplicial categories : 2 versions

strict ones:
$\partial_i\partial_j = \partial_{j-1}\partial_i$
Cat as category

Will use this

lax ones:
$\partial_i\partial_j \xrightarrow{\sim} \partial_{j-1}\partial_i$
Cat as 2-cat
Known as pseudo-simplicial

<u>Waldhausen $S$-construction</u> $\mathcal{A}$ abelian category

$S_\bullet(\mathcal{A}) = \{ S_n(\mathcal{A}), \partial_i, s_i \}$ simplicial groupoid

$S_0(\mathcal{A}) = pt$

$S_1(\mathcal{A}) = \mathcal{A}^\sim \qquad S_2(\mathcal{A}) = SES(\mathcal{A})$

$S_n(\mathcal{A})$ = $\overset{\text{(groupoid)}}{\text{category}}$ of diagrams $\Sigma$

$A_{ij} \in \mathcal{A} \qquad 0 \le i < j \le n$

+ morphisms transitive $\quad A_{ij} \xrightarrow{\varphi_{ij}^{kl}} A_{kl}$ for $\begin{array}{ccc} i & < & j \\ \rotatebox{-90}{$\le$} & & \rotatebox{-90}{$\le$} \\ k & < & l \end{array}$

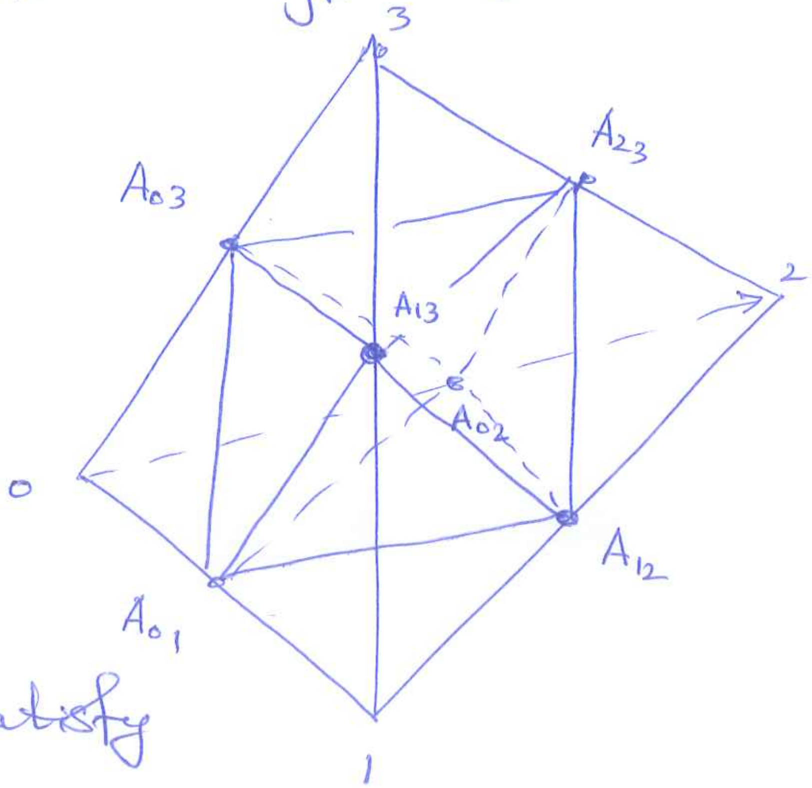
ie. functors $\underset{\text{poset}}{\text{Pairs}\{0,1\ldots n\}} \longrightarrow \mathcal{A}$

such that: $\forall\ i<j<k$ the seq.

$$0 \longrightarrow A_{ij} \xrightarrow{\varphi_{ij}^{ik}} A_{ik} \xrightarrow{\varphi_{ik}^{jk}} A_{jk} \longrightarrow 0$$

is exact

$n=3$ octohedron




$\Sigma$

$\downarrow$ equiv. of groupoids

$\{A_{01} \hookrightarrow A_{02} \hookrightarrow \cdots \hookrightarrow A_{0n}\}$

$\cap$

$\mathrm{Filt}_n(\mathcal{A})$

but $\partial_i : S_n \to S_{n-1}$ satisfy relations strictly

$$\begin{tikzcd}[column sep=small]
A_{01} \arrow[dr] & & A_{12} \arrow[dr] & & A_{23} \arrow[dr] & & A_{34} \\
& A_{02} \arrow[ur] \arrow[dr] & & A_{13} \arrow[ur] \arrow[dr] & & A_{24} \arrow[ur] & \\
& & A_{03} \arrow[ur] \arrow[dr] & & A_{14} \arrow[ur] & & \\
& & & A_{04} \arrow[ur] & & &
\end{tikzcd}$$

<u>2-Segal property</u>

usual fiber product

$X$ simplicial — set / groupoid / top. space $\Big\}$ $\exists$ concept of <u>homotopy</u> fiber product


$$\begin{tikzcd}
Z_1 \arrow[dr, "p_1"'] & & Z_2 \arrow[dl, "p_2"] \\
& W &
\end{tikzcd}$$

$z_1, z_2$ + path joining $p_1(z_1), p_2(z_2)$

More generally, homotopy limit

$\underleftarrow{\text{holim}}\, Z_\alpha$ not just paths but mapped simplices

$T$ a simpl. set

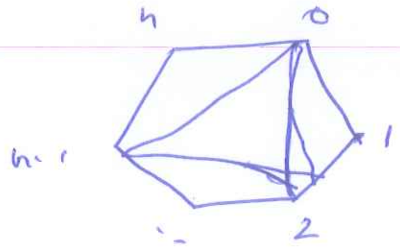
$X_T = \text{Map}^h(T, X)$ — set / groupoid / space $\quad \underset{\sigma \in T}{\underleftarrow{\text{holim}}}\, X_{\dim(\sigma)}$

Say $T = \begin{tikzcd} \bullet \arrow[r, bend left, "\sigma_1"] \arrow[r, bend right, "\sigma_2"'] & \bullet \end{tikzcd}$

$$X_T = \underset{}{\text{holim}} \left( \begin{tikzcd} X_1 \arrow[r] \arrow[dr] & X_0 \\ X_1 \arrow[r] \arrow[ur] & X_0 \end{tikzcd} \right)$$

Membrane space

$P_{n+1} = (n+1)$ gon
vert. numbered

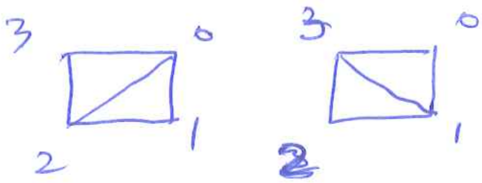



$T$ triangulation

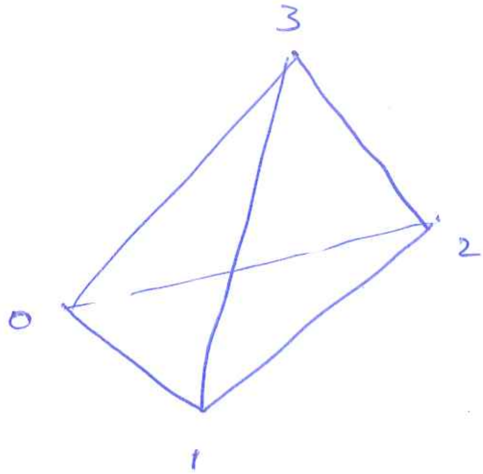
$$\varphi_T : P_{n+1} \hookrightarrow \Delta^n$$

any triangle $\xrightarrow{1:1}$ corr. 2-face of $\Delta^n$

~~Well known lemma:~~




back    front




$$\sigma_T : X_n = X_{\Delta^n} \longrightarrow X_T$$

<u>Def.</u> $X$ is called 2-Segal if for each $n$ and each triangulation $T$ of $P_n$ the map $\sigma_T$ is an equivalence.

<u>Th.</u> For any abelian cat. $\mathcal{A}$ the simpl. groupoid $S_\cdot(\mathcal{A})$ is 2-Segal

Toen: derived $H(A)$

A formalism
$$\begin{array}{ccc} X_n & \mapsto & \mathcal{F}(X_n) \\ \varphi \downarrow & & \varphi^* \uparrow\downarrow \varphi_* \\ Y & \mapsto & \mathcal{F}(Y) \end{array}$$

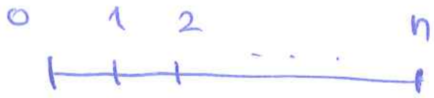
gives a Hall algebra $H(X)$, $X$ 2-Segal space with $X_0 = pt$

## 2-Segal and 1-Segal

X simplicial set, Grothendieck asked: when $X = N\mathcal{C}$ for some cat. $\mathcal{C}$?

$$X_n = \{ x_0 \to x_1 \to \cdots \to x_n \}$$

system of 1-simplices



<u>Answer</u>: when for any "triangulation" T of $[0,n]$ with vertices in $\{0,1,\dots,n\}$ the map

$$X_n \longrightarrow X_T \quad \text{is a bijection} \qquad T = 0 \le i_1 < \dots < i_p \le n$$

enough: $X_n = X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1$ (for T = triang. into unit intervals)

This is called (1-)Segal: <u>1-Segal simpl sets $\leftrightarrow$ categories</u>

also for simpl. spaces (then $X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$)

<u>Th</u>: 1-Segal <u>simpl. spaces</u> $\longleftrightarrow$ $(\infty,1)$-categories [Lurie, Rezk, ...]

<u>Prop</u>. 1-Segal $\Rightarrow$ 2-Segal

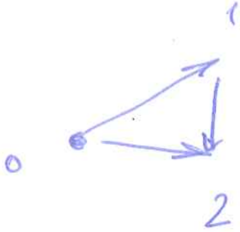
Path space criterion  $X$ simpl. ~~object~~ set

$P^{\triangleleft}X$ left path space  [Illusie]

$(P^{\triangleleft}X)_0 = X_1$

new vertex

$(P^{\triangleleft}X)_1 = X_2$




edge path from $[0,1]$ to $[0,2]$

- -

...

$\Delta \hookrightarrow \Delta$

finite ordinals

$I \longmapsto * \sqcup I$

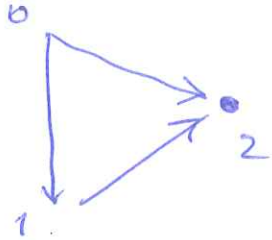
$* < I$

$P^{\triangleright}X$ right path space

$(P^{\triangleleft}X)_0 = X_1$

$(P^{\triangleright}X)_1 = X_2$

...




path from $[0,2]$ to $[1,2]$

$\Delta \hookrightarrow \Delta$

$I \longmapsto I \sqcup * \quad I < *$

Make sense for any simpl. object $X \in \Delta^{\circ}\mathcal{C}$

Th If $X \in \Delta^{\circ}\mathcal{C}$ is such that

$P^{\triangleleft}X$ and $P^{\triangleright}X$ are 1-Segal,

then $X$ is 2-Segal

## Further examples of 2-Segal simplicial sets

(1) Cyclic nerve of a category $\mathcal{C}$

$$NC_n(\mathcal{C}) = \left\{ \begin{tikzcd} x_0 \arrow[r] & x_1 \arrow[r] & \cdots \arrow[r] & x_n \arrow[lll, bend left] \end{tikzcd} \right\}$$

(1') The factorization nerve. R ring ~~R~~ $w \in$ Center(R).

Matrix fact $M_0 \underset{\phi}{\overset{d}{\rightleftarrows}} M_1$, $d^2 = w$

(2) Bruhat-Tits building

$k$ field $K = k((t)) \supset \mathcal{O} = k[[t]]$

Laurent Taylor

$V/K$ n-dim space

$\Gamma(V) = \{ \mathcal{O}\text{-lattices in } V \}$

$\mathcal{O}$-subm. $L \subset$

$K \otimes L \cong V$ e.g. $V = K^n$

$L = \mathcal{O}^n$

or $t^{d_1}\mathcal{O} \oplus \cdots \oplus t^{d_n}\mathcal{O}$

$BT(V)$ simpl. set

$$BT_n(V) = \{ (\underbrace{L_0 \subset L_1 \subset \cdots \subset L_n}_{\substack{\text{just this would give} \\ N(\Gamma(V), \leq) \\ 1\text{-Segal}}} \subset t^{-1} L_0) \mid L_i \in \Gamma(V) \}$$

$BT(V)$ is 2-Segal

...

## Simplicial objects and cyclic objects

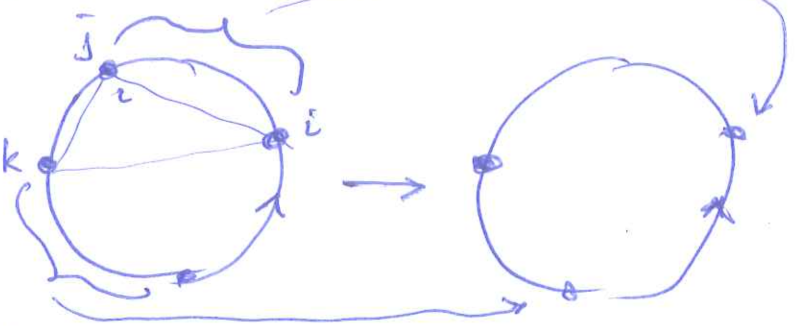
$$[n] \longmapsto X_n$$

$\emptyset \neq$ finite ordinals $(I, \leq)$ $\quad \Delta \xrightarrow{X} \mathcal{E}$ simpl. object

$\cap$

finite $\neq \emptyset$ "cyclic ordinals" $\quad \Lambda \nearrow$ cyclic object is such functor

I with cyclic order $= I \hookrightarrow S^1$ mod isotopy




oriented circle

ternary relation

$$\mathrm{Hom}(I \hookrightarrow S^1, J \hookrightarrow S^1) =$$

$$= \pi_0 \left\{ \text{nonstrictly monotone } \underline{\underline{\text{degree } 1}} \text{ maps } (S^1, I) \to (S^1, J) \right\}$$

[Connes]

$$\langle n \rangle = \{(n+1)\text{th roots of } 1\}$$

Cyclic obj $\quad X =$ simpl. object $+$

$$\tau_n : X_n \to X_n \quad \tau_n^{n+1} = \mathrm{Id}$$

with relations

Ex Cyclic nerve $NC(\mathcal{C})$ is a cyclic set

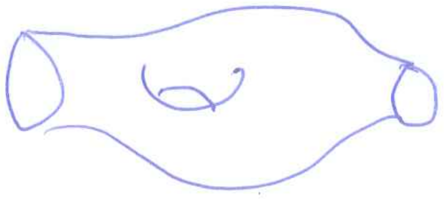
Interesting: 2-Segal cyclic ~~obj~~ spaces

↑

as simplicial ones

# Triangulation complex of a marked surface

$(S,M)$ $\quad$ $S$ compact, oriented top. surface possibly with $\partial S$

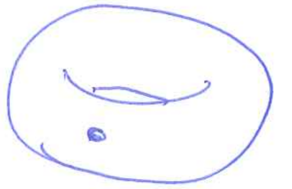


$\emptyset \neq M \subset S$ finite s.t.:

1) $\forall$ component of $\partial S$ (if any) has at least one pt. of $M$

2) $\chi(S,M) = \chi(S) - |M| < 0$ $\qquad$ ($\Sigma_g$, one pt) OK

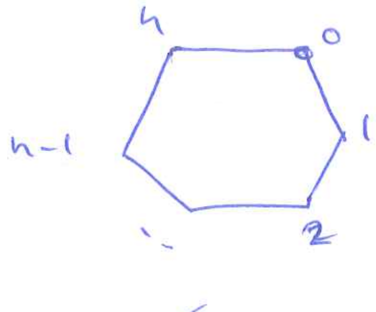
Ex 1)



$S = \mathbb{T}^2$, $M = \{0\}$

~~2) |∂S|=1, |M|=3 pts~~

2) $S = P_{n+1}$ $(n+1)$-gon, $M = \{\text{vertices}\}$




...

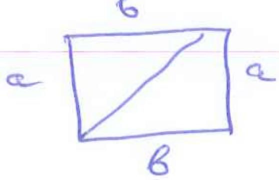
$K_{S,M}$ ("surface associahedron") CW-complex

Vert = curved triangulations of $S$ with vertices in $M$ / isotopy

$\Pi$ Stasheffs = ~~Edges~~ (cells) faces $\Longleftrightarrow$ curved polygonal decompositions / isot

e.g. edges = flips of curved triang

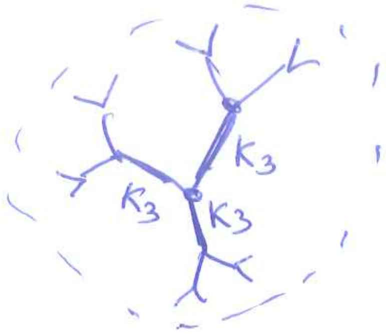
Ex. $K_{\mathbb{T}^2,0} = ?$




Standard triang
3 edges.

Can flip any of 3

infinite tree




Fund. fact ("Teichmüller theory") $K_{S,M}$ is contractible (typically not a convex polytope)

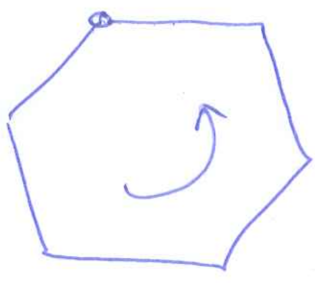
Surface Membrane spaces assoc to 2-Segal cyclic spaces

X 2-Segal cyclic — set / grpd / space

I cyclically ord. set $\Rightarrow$ $X_I$ makes sense

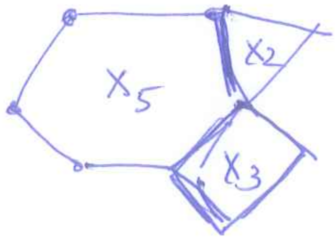
! $P \subset S$ curved polygon (~~vertices~~ vertices possibly coinciding)

$\Rightarrow$ Ed(P) cyclically ordered (by orient. of S)



$T$ triang. of $(S, M)$ or polyg. decomp. $\to$ can form $X_T$ set / grpd / space

$$\cdots X_3 \overset{h}{\underset{X_1}{\times}} X_5 \overset{h}{\underset{X_1}{\times}} X_2 \cdots$$




numbering of vertices not important

$=$ "Map" $(T, X)$ Membrane space.

<u>Th</u> $X_T$ canonically (coherently) independent of $T$ (up to equiv.)

<u>Pf</u> Contractibility of $K_{S,M}$

So we have a canonical $X_{S,M} = \text{any } X_T$

## Making $S$-construction cyclic: derived category

$S_2(\mathcal{A}) = SES(\mathcal{A})$ $\qquad 0 \to A \to C \to B \to 0$

no $\mathbb{Z}/3$ action

$\mathcal{E} = \mathcal{D}^b(\mathcal{A})$ derived cat. $= \mathrm{Compl}(\mathcal{A})[\mathrm{qis}^{-1}]$

$A^\bullet = \{ \cdots \to A^i \xrightarrow{d_i} A^{i+1} \xrightarrow{d_{i+1}} \cdots \}$, $d^2 = 0$

$\varphi : A^\bullet \to B^\bullet$ qis if

$\varphi_* : H^i(A) \underset{\text{iso}}{\xrightarrow{\sim}} H^i(B) \quad \forall i$

$H^i(A^\bullet) = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$

$(\Sigma A)^i = A^{i+1}$ — shifted complex

$\mathcal{A} \subset \mathcal{D}^b(\mathcal{A}) = \mathcal{E}$

$\exists$ class of exact triangles

1) Any SES in $\mathcal{A}$ gives an exact tr. in $\mathcal{E}$

$$\begin{tikzcd} \Sigma(A) & A^\bullet \arrow[r] & C^\bullet \arrow[dl] \\ B & B^\bullet \arrow[u, "+1"] & \end{tikzcd}$$

2) $\frac{2\pi}{3}$ rotation of an exact tr. is exact tr!

$$\underbrace{\Sigma^{-1} C \xrightarrow{\Sigma^{-1}\gamma} A \xrightarrow{\alpha} B}_{\tau_2(\Delta)\text{ also exact}} \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma(A)$$

($A \to B \to C \to \Sigma(A)$: exact tr. $\Delta$)

3 times rotation gives

$$\Sigma^{-2} A \to \Sigma^{-2} B \to \Sigma^{-2} C \to \Sigma^{-1}(A)$$

$(-2)$-shifted triangle $\qquad \tau_2^3 = \Sigma^{-2}$, not Id

<u>2-periodic</u> deriv. cat. $\mathcal{D}^{(2)}(\mathcal{A})$

$$\begin{tikzcd} A^{\bar 0} \arrow[r, bend left, "d"] & A^{\bar 1} \arrow[l, bend left, "d"] \end{tikzcd}$$

$A^\circ$ $\quad A^i = A^{i+2}$

$d_i = d_{i+2}$

Then $\tau_2^3 = \mathrm{Id}$.

$\exists$ <u>dg-enhancement</u> for $\mathcal{D}(\mathcal{A})$

<u>$\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$</u> : complex $= \mathrm{Hom}^\bullet(P_{A^\bullet}, P_{B^\bullet})$

proj. resol.

$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B) = H^0$ of this compl

<u>dg-categories</u> $\mathcal{E}$ : $\forall$ $\mathrm{Hom}_{\mathcal{E}}(x, y)$

made into a complex of k-vect. spaces

in a comp. way

$H^\circ(\mathcal{E})$ : new category, $\mathrm{Hom} = H^\circ(\mathrm{Hom}_{\mathcal{E}})$

Can work with dg-cat. like with spaces: quasi-equivalences

homot. fiber products

- - - .

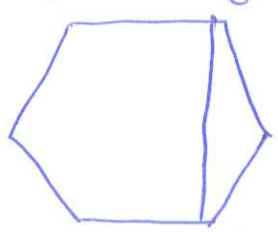
Can speak about <u>2-Segal</u> simplicial or cyclic dg-categories.

## S-construction for dg-categories

k field $\mathrm{Rep}_k(A_n)$ $\quad 1 \to 2 \to \cdots \xrightarrow{\ n-1\ } n$

$k_{[i,j]}$ $1 \leq i \leq j \leq n$ indecompos

$i \mapsto i-1$ labels $0 \leq i < j \leq n$




= chords $C$ in $P_{n+1}$ (including sides)

$k_C$ : corr. repr.

$\mathcal{D}^{(2)}\mathrm{Rep}(A_n)$ "root category"

indec: $k_C$ and $\Sigma k_C$ $\qquad \Sigma^2 = Id$

$j \to i$

$i \to j$ label: oriented chords $C$

Geometrically 1) Hom or $\mathrm{Ext}^1(k_C, k_{C'}) \neq 0$ only if $C \cap C' \neq \emptyset$

2) SES or exact triangles: —


Coxeter functor

This is all $\mathbb{Z}/_{n+1}$ - invariant

$\mathbb{Z}/_{n+1}$ acts on $\mathcal{D}^{(2)}\mathrm{Rep}(A_n)$

Th. $\exists$ dg-categories $S_n \sim \mathcal{D}^2\mathrm{Rep}(A_n)$ which form a co-cyclic object in dg-cat.

<u>NB</u> An exact triangle in $\mathcal{E} = \mathcal{D}^b(\mathcal{A})$

$\updownarrow$

a dg-functor $\mathcal{D}^b \mathrm{Rep}(A_2) \longrightarrow \mathcal{E}$

$\mathcal{D}^{(2)}$ if $\mathcal{E}$ 2-per.

Now $\mathcal{E}$ any 2-periodic dg-Cat

$$S_n(\mathcal{E}) = \mathrm{dgfun}(S_n, \mathcal{E})$$

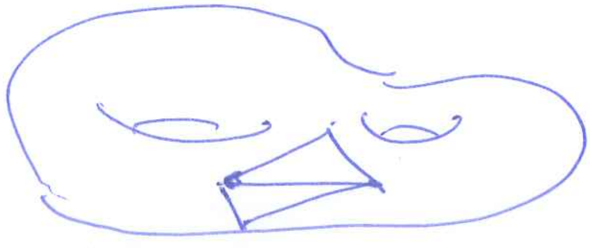
<u>Th</u> These form a <u>2-Segal</u> cyclic dg-cat.

<u>Cor</u>. To every marked surface $(S, M)$ one can associate a dg-cat

$$\mathcal{F}(S, M; \mathcal{E}) = S(\mathcal{E})_{S,M}$$

= topological Fukaya cat.

analog of $H_1(S, M; \underset{\text{ab. gr.}}{E})$



objects: $T$-systems of exact triangles

$T$ any triang.