III. LATTICES

Aim: By construction, the relations R_I are positive (no I_α^{-1}) and, for all α, β, there is (at most) one relation $I_\alpha \ldots = I_\beta \ldots$
Exploit this to investigate lattice properties of the monoid $\langle \{I_\alpha | \alpha \in \mathbb{A}\} | R_I \rangle^+$.

1. Complemented presentations

We have seen $F = \langle \{A_\alpha | \alpha \in \mathbb{A}\} | R_A \rangle$

$$F^+ := \langle \{A_\alpha | \alpha \in \mathbb{A}\} | R_A \rangle^+$$

monoid

NB: Write α for A_α

Question: Connection between F^+ and F?
Does F^+ embed in F?

Fact: $\langle \mathbb{A} | R_A \rangle^+$ is a complemented presentation

every relation of the type $u = v$ with $u, v \neq \varepsilon$
and $\forall \alpha, \beta \in \mathbb{A}$ $\exists!$ relation $(\alpha | \ldots = \beta | \ldots)$

<u>Def.</u> Assume $(S\,|\,R)$ right complemented. For $u, v, u', v' \in S^*$

$(u, v) \curvearrowright_R (u', v')$ "right reverses"

if $\exists$ finite grid diagram
$$\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\curvearrowright"] & \cdot \arrow[d, "u'"] \\ \cdot \arrow[r, "v'"'] & \cdot \end{tikzcd}$$

with squares of the form
$$\begin{tikzcd} \cdot \arrow[r, "t"] \arrow[d, "s"'] & \cdot \arrow[d, "C(t,s)"] \\ \cdot \arrow[r, "C(s,t)"'] & \cdot \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, "s"'] & \cdot \arrow[d, equal] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

presentation $sC(s,t) = tC(t,s)$.

<u>Example</u> (case of F^+)

$u = 1|10$ (i.e., $A_1 A_{10}$) $\quad v = \emptyset|01$ (i.e., $A_\emptyset A_{01}$)

$$\begin{tikzcd} \cdot \arrow[rrr, "\emptyset"] \arrow[d, "1"'] & & & \cdot \arrow[r, "01"] \arrow[d, "\emptyset"] & \cdot \arrow[d, "\emptyset"] \\ \cdot \arrow[r, "\emptyset"] \arrow[d, "10"'] & \cdot \arrow[rr, "0"] \arrow[d, "01"] & & \cdot \arrow[r, "001"] \arrow[d, "0"] & \cdot \arrow[d, "0"] \\ \cdot \arrow[r, "\emptyset"'] & \cdot \arrow[r, "0"'] & \cdot \arrow[r, "00"'] & \cdot \arrow[r, "000"'] & \cdot \end{tikzcd}$$

$(1|10, \emptyset|01) \curvearrowright_{RA} (\emptyset|0, \emptyset|0|00|000)$.

* By def. $(u,v) \curvearrowright_R (u',v')$ implies $uv' \equiv_R vu'$

$\rightsquigarrow$ witness for a common right-multiple of $[u]$ and $[v]$ in $\langle S\,|\,R\rangle^+$.

<u>Proposition 1</u> Assume (S,R) right-complete and
(i) $\exists \lambda : S^* \to \mathbb{N}$ (λ $\equiv_R$-invariant and
$\forall s \in S\ \forall u \in S^* (\lambda(su) > \lambda(u))$);
(ii) $\forall r,s,t \in S$, if $\exists u', v'$
$$\left(\begin{tikzcd} \cdot \arrow[r, "s\,C(s,t)"] \arrow[d, "r"'] \arrow[dr, phantom, "\curvearrowright_R"] & \cdot \arrow[d, "u'"] \\ \cdot \arrow[r, "v'"'] & \cdot \end{tikzcd}\right)$$
then $\exists u'_1, v'_1$
$$\left(\begin{tikzcd} \cdot \arrow[r, "t\,C(t,s)"] \arrow[d, "r"'] \arrow[dr, phantom, "\curvearrowright_R"] & \cdot \arrow[d, "u'_1"] \\ \cdot \arrow[r, "v'_1"'] & \cdot \end{tikzcd} \text{ with } u'_1 \equiv_R u',\ v'_1 \equiv_R v'\right).$$
Then $u \equiv_R v$ is equivalent to $(u,v) \curvearrowright_R (\varepsilon, \varepsilon)$.

<u>Proof.</u> Prove using induction on $\lambda(uv')$ that
if $\exists u', v'$
$$\left(\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\curvearrowright_R"] & \cdot \arrow[d, "u'"] \\ \cdot \arrow[r, "v'"'] & \cdot \end{tikzcd}\right)$$
and $u_1 \equiv_R u$, $v_1 \equiv_R v$,
then $\exists u'_1, v'_1$
$$\left(\begin{tikzcd} \cdot \arrow[r, "v_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\curvearrowright"] & \cdot \arrow[d, "u'_1"] \\ \cdot \arrow[r, "v'_1"'] & \cdot \end{tikzcd} \text{ with } u'_1 \equiv_R u',\ v'_1 \equiv_R v'\right).$$
Then $(u,u) \curvearrowright_R (\varepsilon, \varepsilon)$ (always), hence
$u \equiv_R v$ implies $\exists u'_1, v'_1$
$$\left(\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"'] & \cdot \arrow[d, "u'_1"] \\ \cdot \arrow[r, "v'_1"'] & \cdot \end{tikzcd} \text{ with } u'_1 \equiv_R \varepsilon,\ v'_1 \equiv_R \varepsilon\right)$$
implies $u'_1 = v'_1 = \varepsilon$. ∎

Coro. Under the assumptions of Prop 1

(i) $\langle S|R\rangle^+$ is left-cancellative

(ii) $\langle S|R\rangle^+$ admits conditional right lcms, namely for all u, v in S^* TFAE

* $[u], [v]$ admit a common right multiple in $\langle S|R\rangle^+$
* ——— right lcm ———
* $\exists u', v' \, ((u,v) \curvearrowright (u',v'))$.

(iii) $\langle S|R\rangle^+$ admits left-gcds.

Proof. (i) Assume $su \equiv_R sv$.
Then $(su, sv) \curvearrowright_R (\varepsilon, \varepsilon)$
Hence $(u, v) \curvearrowright_R (\varepsilon, \varepsilon)$, hence $u \equiv_R v$.

(ii) Assume $uv_1 \equiv_R vu_1$. Then $(uv_1, vu_1) \curvearrowright (\varepsilon, \varepsilon)$

$$\begin{tikzcd}
\cdot \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\curvearrowright_R"] & \cdot \arrow[r, "u_1"] \arrow[d, "u'"] & \cdot \arrow[dd, "\varepsilon"] \\
\cdot \arrow[r, "v'"'] \arrow[d, "v_1"'] & \cdot \arrow[r] \arrow[d] & {} \\
\cdot \arrow[r] & \cdot \arrow[r, "\varepsilon"'] & \cdot
\end{tikzcd}$$

Then $[uv']$ common right-multiple of $[u], [v]$ and $[uv_1]$ right multiple of $[uv']$.
↑
right-lcm of $[u], [v]$
☐

(ii) $\Rightarrow$ (iii).

* Applies to $F^+ = \langle A \mid R_A \rangle^+$

⇝ $\lambda(w) := \#_{0\text{-edges}}(\underline{\text{skel}}(A_w^{L})) - \#_{0\text{-edges}}(\underline{\text{skel}}(A_w^{e}))$

⇝ * <u>hard</u> case *:

$$\begin{tikzcd}
{} \arrow[r, "\emptyset"] \arrow[d, "\shortparallel"'] & {} \arrow[rr, "\emptyset"] \arrow[d, "1"] & & {} \arrow[d, "\emptyset"] \\
{} \arrow[r, "\emptyset"'] & {} \arrow[r, "\emptyset"'] & {} \arrow[r, "0"'] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[rr, "1"] \arrow[d, "\shortparallel"'] & & {} \arrow[rr, "\emptyset"] \arrow[d, "1"] & & {} \arrow[r, "0"] \arrow[d, "\emptyset"] & {} \arrow[d, "\emptyset"] \\
{} \arrow[r, "1"'] & {} \arrow[r, "10"'] & {} \arrow[r, "\emptyset"'] & {} \arrow[r, "0"'] & {} \arrow[r, "00"'] & {}
\end{tikzcd}$$

$$\emptyset|\emptyset|0 \equiv_{R_A} 1|\emptyset|0|0 \equiv_{R_A} 1|\emptyset|01|0|00 \equiv_{R_A} 1|10|\emptyset|0|00$$

<u>Proposition</u>.
- F^+ is cancellative
- F^+ admits conditional lcms
- F^+ admits gcds

<u>Prop 2</u>: Assume (S, R) right complemented and
(iii) $\exists \hat{S} \subseteq S^* (S \subseteq \hat{S}$ and
$\forall u, v \in \hat{S}\ \exists u', v' \in \hat{S}\ ((u,v) \curvearrowright_R (u', v')))$.
Then $\forall u, v \in S^*\ \exists u', v' \in S^* ((u,v) \curvearrowright_R (u',v'))$

<u>Coro</u>: Under the assumptions of Prop 2
$\langle S \mid R \rangle^+$ admits common right-multiples.

* Applies to $F^+ = \langle A \mid R_A \rangle^+$
with $\hat{A} := \{\alpha | \alpha 0 | \alpha 00 | \ldots | \alpha 0^p \mid \alpha \in A, p \in \mathbb{N}\}$.

$$\begin{tikzcd}
{} \arrow[r, "1"] \arrow[d, "\emptyset"'] & {} \arrow[d, "\emptyset|0"] \\
{} \arrow[r, "\emptyset"'] & {}
\end{tikzcd}$$

<u>Proof.</u> F^+ admits common multiples, hence lcms.

<u>Coro</u> (i) F^+ embeds in F and F is a group of fractions of F^+.

(ii) $(F^+, \leqslant)$ is a lattice

the left-divisibility relation
$a \leqslant b$ iff $\exists x\ (ax = b)$

(iii) $(F, \leqslant)$ is a lattice
$a \leqslant b$ iff $\exists x \in F^+\ (ax = b)$.

2. Associahedra and Tamari lattices.

For $n \geqslant 1$, $\mathcal{T}_n := \{$ size n trees $\}$ $\#\mathcal{T}_n = \mathrm{Cat}_n$.

size: n leaves

<u>Fact.</u> For every T in $\mathcal{T}_n$

* $T \bullet F = \mathcal{T}_n$

partial action

* in particular: $x^{[n]} \bullet F^+ = \mathcal{T}_n$

partial action

* " $x^{[n]} \bullet \underline{\mathrm{Div}}(\delta^{n-2}) = \mathcal{T}_n$

total action

(Proof $T \bullet g = T \bullet g'$ implies $g = g'$ in F, hence in F^+ for $g, g' \in F^+$ because $F^+ \hookrightarrow F$.)

Prop (Tamari) For T, T' in $\mathcal{T}_n$, declare $T \preccurlyeq T'$ if $\exists g \in F^+ (T \bullet g = T')$.
Then, for each n, $(\mathcal{T}_n, \preccurlyeq)$ is a lattice and the map $\chi : \mathcal{T}_n \hookrightarrow F^+$ establishes an isomorphism between $(\mathcal{T}_n, \preccurlyeq)$ and $(\mathrm{Div}(\phi^{n-2}), \leqslant)$ in F^+

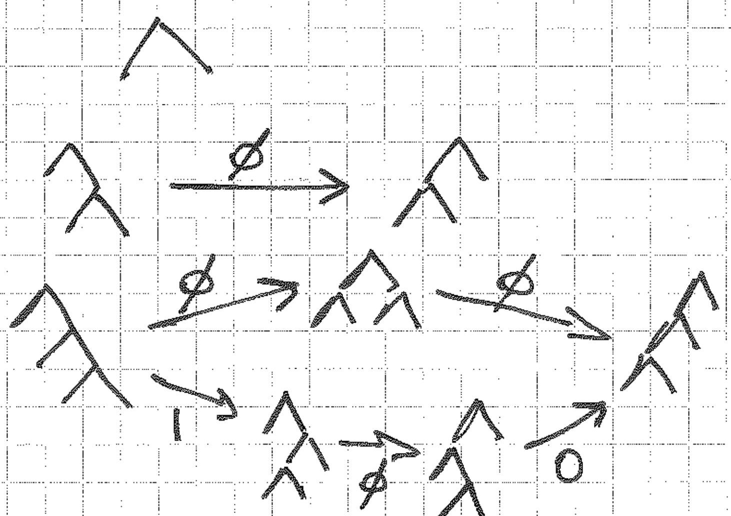
Def. $\mathcal{K}_{n+1}$ ($(n+1)$st associahedron) is the oriented polytope whose 0-skeleton is $\mathcal{T}_n$ and the p-faces starting from T are the sets of the form $\{T \bullet g \mid g \leqslant \mathrm{lcm}(\alpha_1, \dots, \alpha_p)\}$ with $\alpha_1, \dots, \alpha_p$ pairwise $\neq$ in A and $T \bullet \alpha_i \downarrow$ for each i.

[So $\mathcal{K}_{n+1}$ is the restriction of the Cayley graph of F^+ to $\mathcal{T}_n$.

K_2

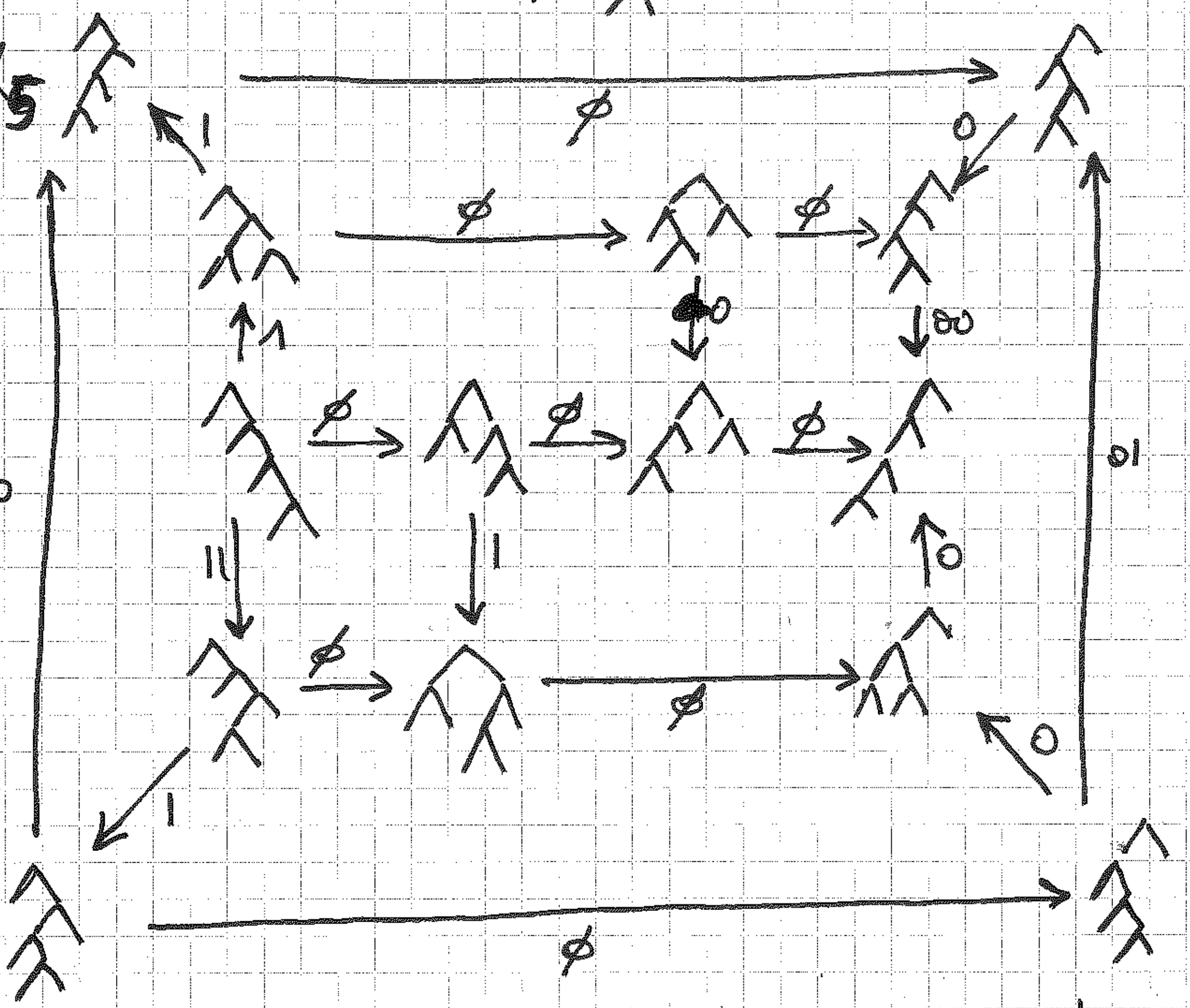
K_3

K_4



5 vertices
5 edges
1 face

K_5

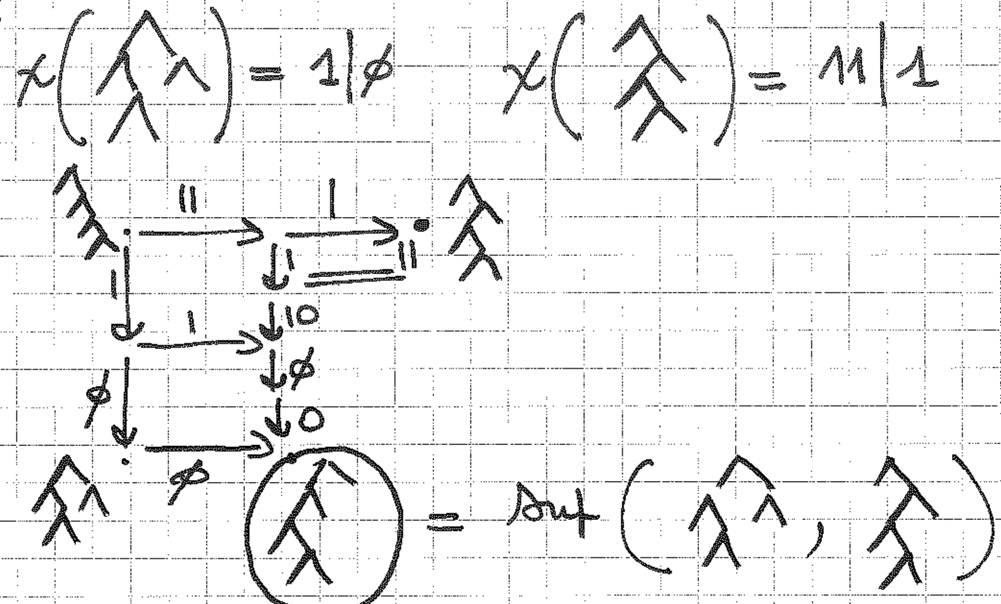


14 vertices
21 edges
9 2-faces
1 3-face.

* How to practically compute $T_1 \underset{\underset{\sup}{\uparrow}}{\vee} T_2$?

$\rightsquigarrow$ evaluate $\chi(T_1)$, $\chi(T_2)$,
then reverse $(\chi(T_1), \chi(T_2))$

Ex:



* Case of (D) : Idem (but) $G_D^+ \hookrightarrow G_D$ unknown

~~...~~ Again G_D^+ has right lcms, (but) unclear whether this provides a lattice structure on terms rel $\leq$: $T \leq T'$ if $\exists g \in G_D^+ (T' = T \cdot g)$.
("Embedding Conjecture").