# II. Présentation & cohérence

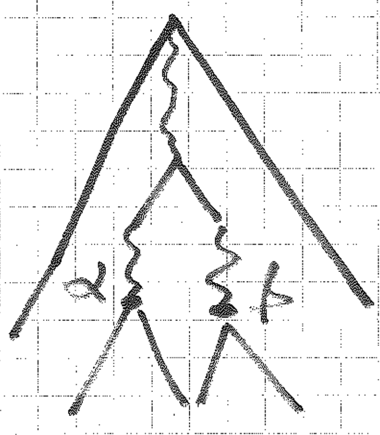
Aim. Find presentation of $G_I$

Step 1: Find relations
Step 2: Prove they make a presentation $\Longleftrightarrow$ coherence

## 1- The relations $R_I$

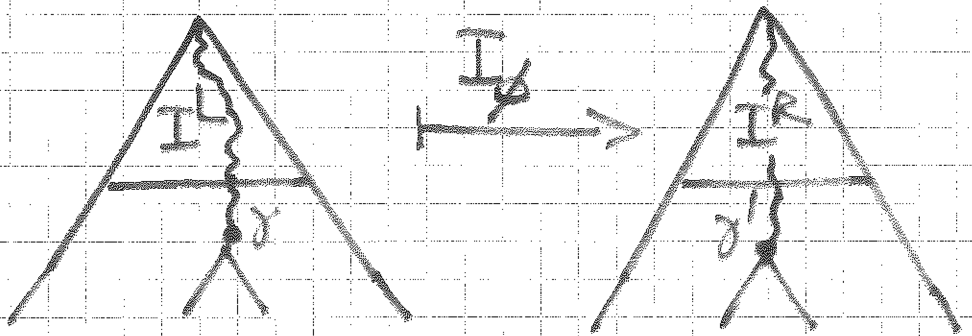
Principle (here): look for $I_\alpha \ldots = I_\beta \ldots$ (confluence)

* (By def) generators $I_\alpha$, $\alpha \in \mathbb{A}$ binary addresses
* Relations, parallel case: $\alpha \perp \beta$



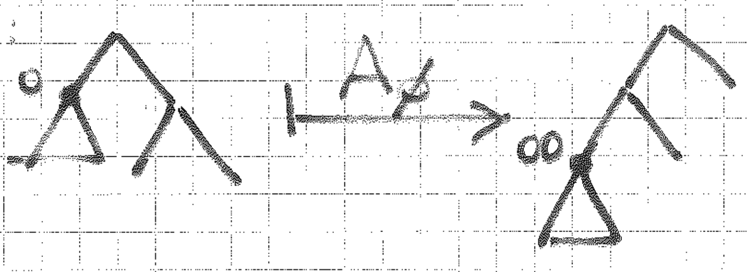
$$I_\alpha \cdot I_\beta = I_\beta \cdot I_\alpha$$

* Relations, <u>nested</u> case $\alpha$, $\alpha\gamma$ with $\gamma \notin$ <u>skel</u>$(I^L)$




$$I_\gamma \cdot I_\phi = I_\phi \cdot I_{\gamma'}$$

<u>Example</u>:




$$A_0 \cdot A_\phi = A_\phi \cdot A_{00}$$

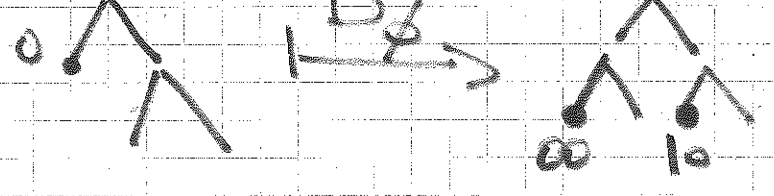
More generally: $A_{0\gamma} \cdot A_\phi = A_\phi \cdot A_{00\gamma}$

" $A_{\alpha 0\gamma} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 00\gamma}$

Similarly: $A_{\alpha 10\gamma} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 01\gamma}$

$A_{\alpha 11\gamma} \cdot A_\alpha = A_\alpha \cdot A_{\alpha 1\gamma}$

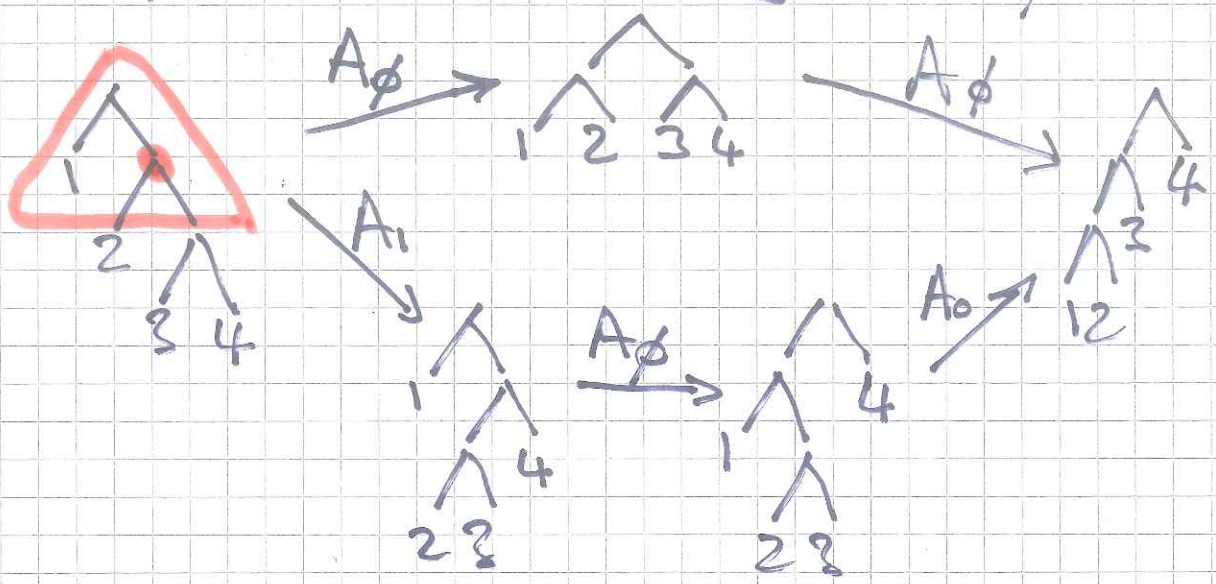
<u>Counter.</u> ? Possible relations?




$$D_0 \cdot D_\phi = D_\phi \cdot D_{00} \cdot D_{10} \ \ldots$$

* Relations, overlapping case: no general method

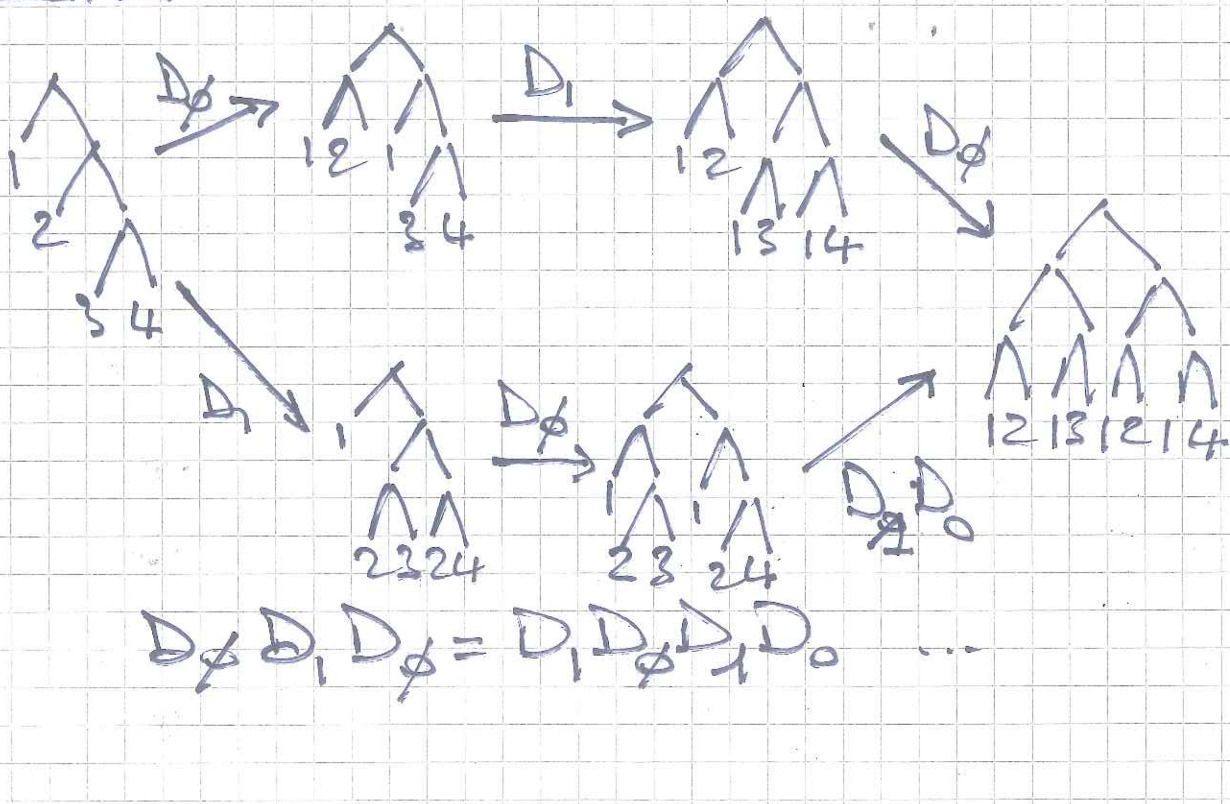
Example (A) one missing case $A_\phi$ vs $A_1$




$$A_\phi \cdot A_\phi = A_0 \cdot A_\phi \cdot A_1$$

More generally $A_\alpha^2 = A_{\alpha 0} \cdot A_\alpha \cdot A_{\alpha 1}$ .

Example: (D) idem




$$D_\phi D_1 D_\phi = D_1 D_\phi D_1 D_0 \;\ldots$$

## 2. The blueprint method

$R_I$ := all confluence relations so obtained
* valid in $G_I$
* conversely, how to prove they present $G_I$?

⇝ (more precisely)

(#) does $I_w \approx I_{w'}$ imply $I_w \equiv_{R_I} I_{w'}$ ?

(if so, $R_I$ presents $G_I/\approx$).

Method. (Assume) $\chi : \underset{\text{terms}}{\mathcal{T}_0} \hookrightarrow \underset{\text{signed addresses}}{(A \cup \bar{A})^*}$

(##) s.t. $I_{\chi(T \cdot I_w)} \equiv_{R_I} I_{\chi(T)} \cdot I_w$

external: action of $G_I$ on $\mathcal{T}$

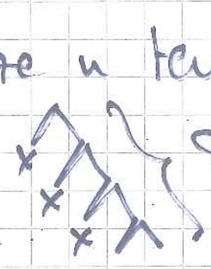
internal in $G_I$ multiplication

Then $I_w \approx I_{w'}$ implies $\exists T (T.w = T.w')$

hence $I_{\chi(T)} I_w \equiv_{R_I} I_{\chi(T)} I_{w'}$

whence $I_w \equiv_{R_I} I_{w'}$

(no hidden relation)

Example: (A) Every size n term is A-equivalent to $x^{[n]}$ :=  size n

→ Define $\chi(T)$ as a distinguished word s.t. $x^{[n]} \xmapsto{A_{\chi(n)}} T$

Then if $T \xmapsto{A_W} T'$,

$$\begin{tikzcd}
 & T \arrow[d, "A_W"] \\
X^{[n]} \arrow[ur, "A_{\chi(T)}"] \arrow[r, "A_{\chi(T')}"'] & T'
\end{tikzcd}$$

$A_{\chi(T)} A_W \approx A_{\chi(T')}$

we can expect $A_{\chi(T)} A_W \equiv_{RA} A_{\chi(T')}$.

$\rightsquigarrow$ checkable from explicit def. of $\chi$.

In fact simultaneously construct $\chi$ and $\chi'$ s.t.

$$A_{\chi(T)} : X^{[n]} \longmapsto T$$

$$A_{\chi'(T)} : X^{[n+p]} \longmapsto T ^\wedge X^{[p]}$$

$\chi(\cdot) = \chi'(\cdot) = \varepsilon$ (empty sequence).

Then for $T = T_0 ^\wedge T_1$

shift by 1

$$X^{[n]} \xmapsto{A_{\chi'(T_0)}} T_0 ^\wedge X^{[n-n_0]} \xmapsto[\cancel{sh(A_{\chi(T_1)})}]{A_{\underline{sh}_1(\chi(T_1))}} T_0 ^\wedge T_1$$

$$X^{[n+p]} \xmapsto{A_{\chi'(T_0)}} T_0 ^\wedge X^{[n+p-n_0]} \xmapsto[A_{\underline{sh}_1(\chi(T_1))}]{\cancel{\underline{sh}_1(A}} T_0 ^\wedge (T_1 ^\wedge X^{[p]}) \xmapsto{A_\phi} (T_0 ^\wedge T_1) ^\wedge X^{[p]}$$

$$\begin{cases} \chi(T_0 ^\wedge T_1) = \chi'(T_0) \mid \underline{sh}_1(\chi(T_1)) \\ \chi'(T_0 ^\wedge T_1) = \chi'(T_0) \mid \underline{sh}_1(\chi'(T_1)) \mid \phi \end{cases}$$

<u>Example</u>: $\chi\left(\begin{smallmatrix}\wedge\\ \wedge\ \wedge\\ \wedge\end{smallmatrix}\right) = \chi'(\wedge) \,|\, \underline{sh}_1\left(\chi\left(\begin{smallmatrix}\wedge\\ \wedge\end{smallmatrix}\right)\right)$

$\chi'(\wedge) = \chi'(\cdot) \,|\, \underline{sh}_1(\chi'(\cdot)) \,|\, \emptyset = \emptyset$

$\chi\left(\begin{smallmatrix}\wedge\\ \wedge\end{smallmatrix}\right) = \chi'(\wedge) \,|\, \underline{sh}_1(\chi(\cdot)) = \emptyset$

$\chi\left(\begin{smallmatrix}\wedge\\ \wedge\\ \ \ \wedge\end{smallmatrix}\right) = \emptyset | 1$

$\chi'\left(\begin{smallmatrix}\wedge\\ \wedge\ \wedge\\ \wedge\end{smallmatrix}\right) = \chi'(\wedge) \,|\, \underline{sh}_1\left(\chi'\left(\begin{smallmatrix}\wedge\\ \wedge\end{smallmatrix}\right)\right) \,|\, \emptyset = \emptyset|1|1|\emptyset$

$\chi'\left(\begin{smallmatrix}\wedge\\ \wedge\end{smallmatrix}\right) = \chi'(\wedge) \,|\, \underline{sh}_1(\chi'(\cdot)) \,|\, \emptyset = \emptyset|\emptyset$

<u>Lemma</u>. For every $T$ and every $\alpha$

$$A_{\chi(T.A_\alpha)} \equiv_{R_A} A_{\chi(T)} \cdot A_\alpha$$

<u>Presentation</u>. $F \; (= \mathcal{G}_A/\approx) = \langle \{A_\alpha \,|\, \alpha \in A\} \,|\, R_A \rangle$

$$\begin{cases} A_\alpha A_\beta = A_\beta A_\alpha & \text{if } \alpha \perp \beta \\ A_\alpha A_{\alpha 00\gamma} = A_{\alpha 0\gamma} A_\alpha \\ A_\alpha A_{\alpha 10\gamma} = A_{\alpha 01\gamma} A_\alpha \\ A_\alpha A_{\alpha 11\gamma} = A_{\alpha 1\gamma} A_\alpha \\ A_\alpha^2 = A_{\alpha 1} A_\alpha A_{\alpha 0} \end{cases}$$

Case of (D)?

* not true that $x^{[n]} \equiv_D T$ for $|T| = n$ ...
* but still $x^{[n]} =_D T^\wedge x^{[n-1]}$ for $n \gg 0$.

$\rightsquigarrow$ define $\chi(T)$ s.t.

$$x^{[n]} \xmapsto{D_{\chi(T)}} T^\wedge x^{[n-1]}$$

Assume $T = T_0{}^\wedge T_1$

* $$x^{[n]} \xmapsto{D_{\chi(T_0)}} T_0{}^\wedge x^{[n-1]} \xmapsto{D_{\underline{sh}_1(\chi(T_1))}} T_0{}^\wedge(T_1{}^\wedge x^{[n-2]})$$

$$\xmapsto{D_\emptyset} (T_0{}^\wedge T_1)^\wedge(T_0{}^\wedge x^{[n-2]}) \xmapsto{\overline{D_{\underline{sh}_1(\chi(T_0))}}} (T_0{}^\wedge T_1)^\wedge x^{[n-1]}$$

If $T' = T \cdot D_w$,

$$\begin{tikzcd}
 & T^\wedge x^{[n]} \arrow[dd, "D_{0w}"] \\
x^{[n]} \arrow[ur, "D_{\chi(T)}"] \arrow[dr, "D_{\chi(T')}"'] & \\
 & T'^\wedge x^{[n]}
\end{tikzcd}$$

<u>Lemma</u>. $D_{\chi(T \cdot D_w)} \equiv_{R_D} D_{\chi(T)} \cdot D_{\underline{sh}_0(w)}$

Then $D_w \approx D_{w'}$ implies $D_{\underline{sh}_0(w)} \equiv_{R_D} D_{\underline{sh}_0(w')}$

$\rightsquigarrow$ from there $D_w \equiv_{R_D} D_{w'}$ but difficult.

Application.

$T \Rightarrow_{D'} T'$ implies $D^{-1}_{\chi(T')} D_{\chi(T)} \in \underline{sh}_0(\mathcal{G}_D)$

Hence, if we collapse $\underline{sh}_0(\mathcal{G}_D)$, then $T$ and $T'$ have the same image in the quotient

Hence, $\underbrace{\mathcal{G}_D/\underline{sh}_0(\mathcal{G}_D)}_{\downarrow \; = B_\infty \text{ (braid group)}}$ is a $D$-algebra.