Associativity, associahedra, and Tamari lattices

<u>Aim</u>. Explain Richard Thompson's construction of the group F as the "geometry group" of associativity, and connect the lattice structure of associahedra with the lattice structure of the monoid F^+.

<u>Plan</u>.

I. The geometry monoid of a law

II. Presentation and coherence

III. The positive monoid and its lattice structure

I. The Geometry monoid of a law

Aim: Associate with every algebraic law I a monoid (group) G_I that captures its geometry (?)

* simple law (e.g., associativity): provide an interesting group (F)
* complicated law (e.g., self-distributivity): helps understanding the law (word pb.)

1. Algebraic laws

An (algebraic) law I is a pair of terms (I^L, I^R) or $I^L = I^R$

terms: parenthesized expressions involving variables and operations

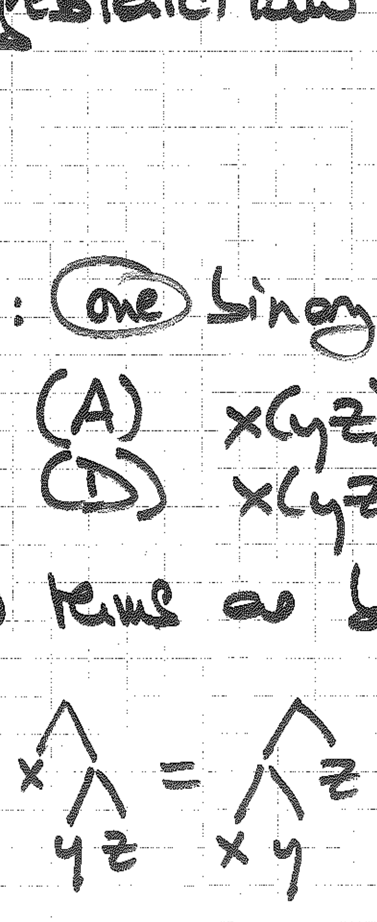
Here: one binary operation

Ex: (A) $x(yz) = (xy)z$

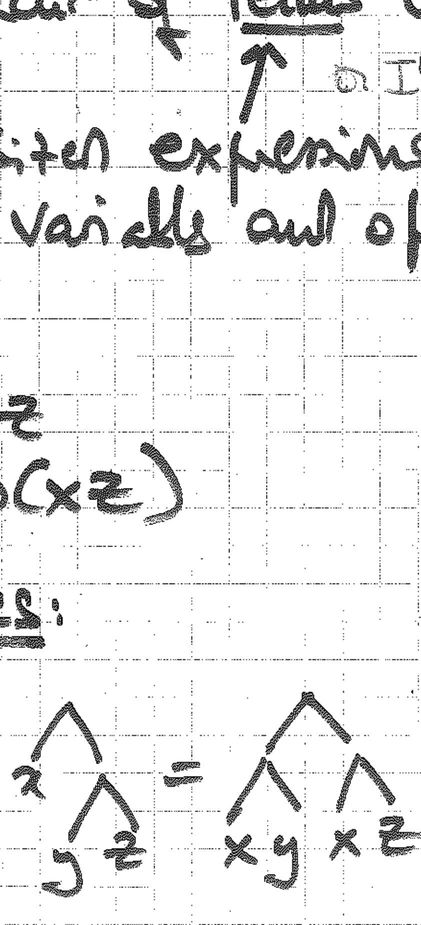
(D) $x(yz) = (xy)(xz)$

View terms as binary trees:

(A)



(D)



Notation.

- $\mathcal{T}_X$ $(\subseteq \mathcal{T})$:= { terms with variables in X }
- $\mathbb{A}$:= { binary addresses } = $\{0,1\}^*$

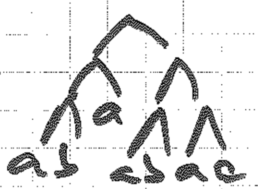
  finite sequences of 0s and 1s
- For $T \in \mathcal{T}$ and $\alpha \in \mathbb{A}$ small enough

  $T/_\alpha$:= αth subterm of T

  Example: $T =$  $T/_0 =$  $T/_{00} = x$ $T/_{000}$ not def'd
- For $T \in \mathcal{T}_X$ and $\sigma : X \to \mathcal{T}_Y$:

  $T\sigma$:= result of substituting x with $\sigma(x)$ in T

  Example: $\sigma(x) =$  $\sigma(y) = a$ $\sigma(z) =$ 

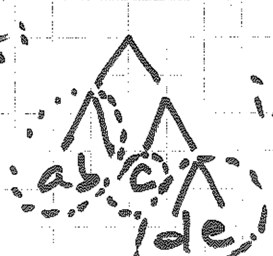
  $T\sigma =$ 

- For I a law and X a set of variables:

$=_{I,X}$:= congruence on $\mathcal{T}_X$ gen'd by all X-instances of I

pairs $(I^L\sigma, I^R\sigma)$ with $\sigma : \mathbb{N} \to \mathcal{T}_X$

Example:

 is an $\{a,b,c,d,e\}$-instance of A

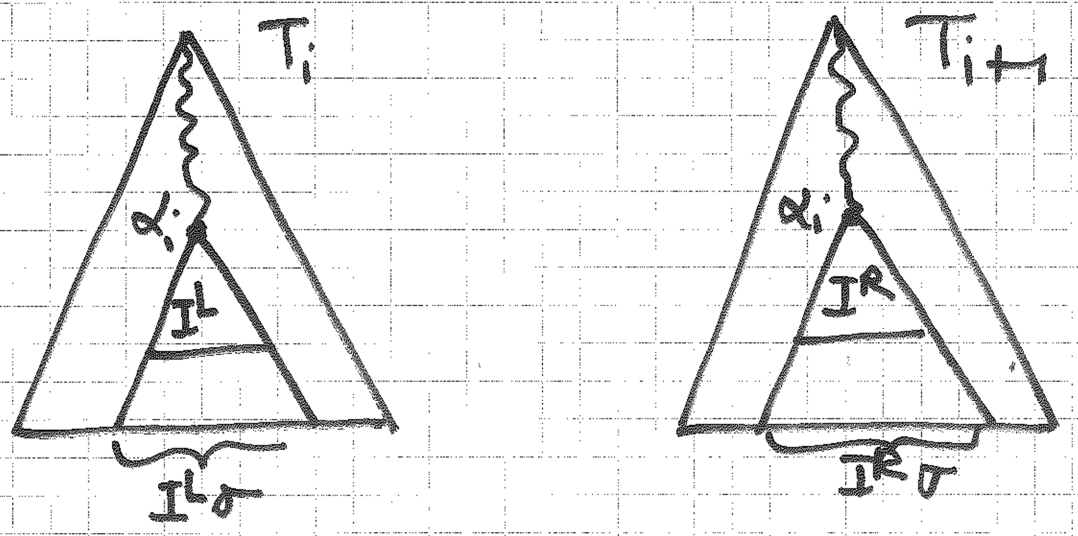
$\sigma(x) =$  $\sigma(y) = c$ $\sigma(z) =$ 

<u>Lemma 2.</u> For every law I

(i) $\mathcal{T}_X / =_{I,X}$ is a free I-algebra gen'd by X

(ii) Two terms T, T' are $=_{I,X}$-equivalent iff there exists a finite sequence $T_0 = T, \dots, T_n = T'$ s.t., for every i, there is an address α_i and a sign e_i $(= \pm)$ s.t.

- $(T_i/\alpha_i \,,\, T_{i+1}/\alpha_i)$ is an instance of I^{e_i}
- $T_i/\gamma = T_i/\gamma$ for $\gamma \perp \alpha$.

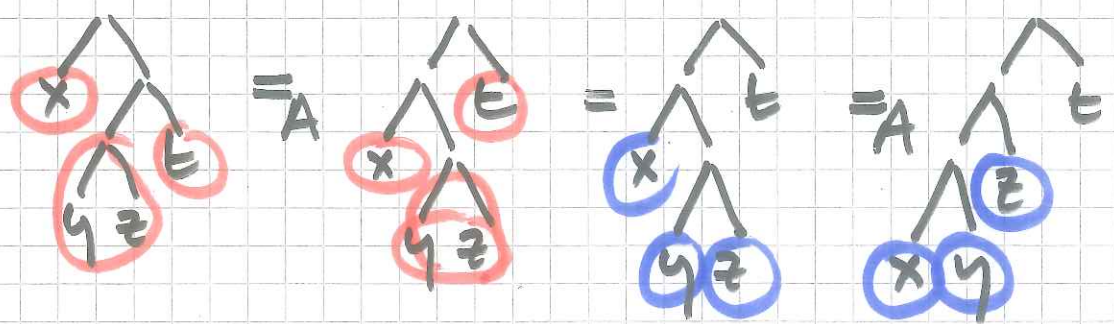


<u>Proof:</u> (i) Let $\mathcal{S}$ be an I-algebra and $F : X \to \mathcal{S}$

$$\begin{tikzcd} \mathcal{T}_X \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{S} \\ \mathcal{T}_X / =_{I,X} \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

(ii) Write $\equiv$ for "$\exists$ sequence ....".
Then $\equiv$ implies $=_{I,X}$. Conversely $\equiv$ is a congruence that contains all X-instances... ∎

Example. $x((yz)t) =_A ((xy)z)t$



2. The geometry monoid $\mathcal{G}_I$

View applying I as a monoid action.

Def. For I a law, α an address, e a sign

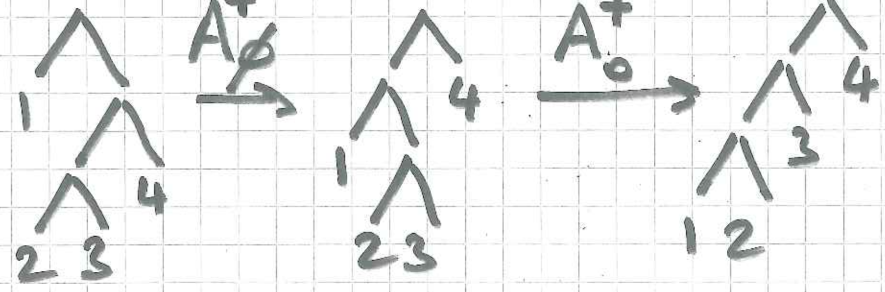
I^e_α := partial map on $\mathcal{T}$ "apply I^e at pos. α"

$\mathcal{G}_I$:= monoid gen'd by all I^e_α.

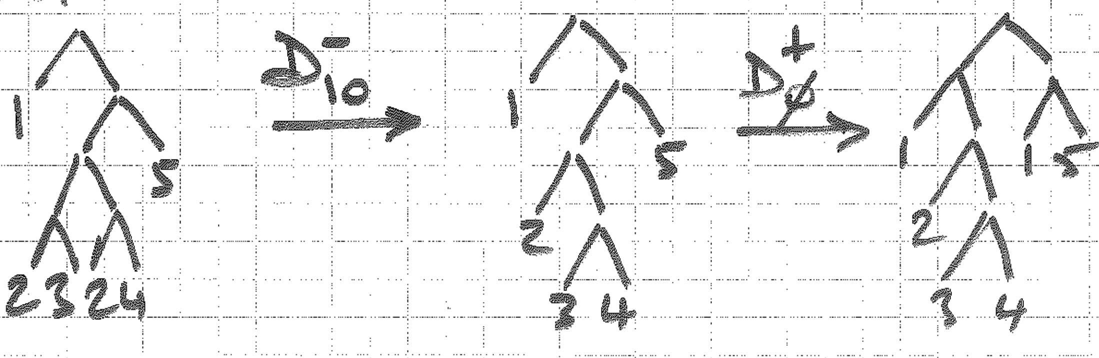
↑ "geometry monoid of I"

Thompson's monoid of I

Example



Example.



Proposition. For every law I, TFAE
(i) $T =_{I,(x)} T'$ holds,
(ii) some element of $\mathcal{G}_I$ maps T to T'.

Remarks. * Partial maps: cannot apply I_α to T when skel(T) is too small, and

skeleton $:= \{\alpha \mid T/_\alpha \text{ exists}\}$

in case of variable clashes.

* Could use categories: objects = terms, morphisms = triples (T, g, T').

Notation. For w a sequence of signed addresses, say $w = \alpha_1^{e_1}|\ldots|\alpha_n^{e_n}$, $I_w := I_{\alpha_1}^{e_1} \ldots I_{\alpha_n}^{e_n}$
(reversed composition)

Example: $D_{10|\emptyset}$, $A_{\emptyset|0}$ ....

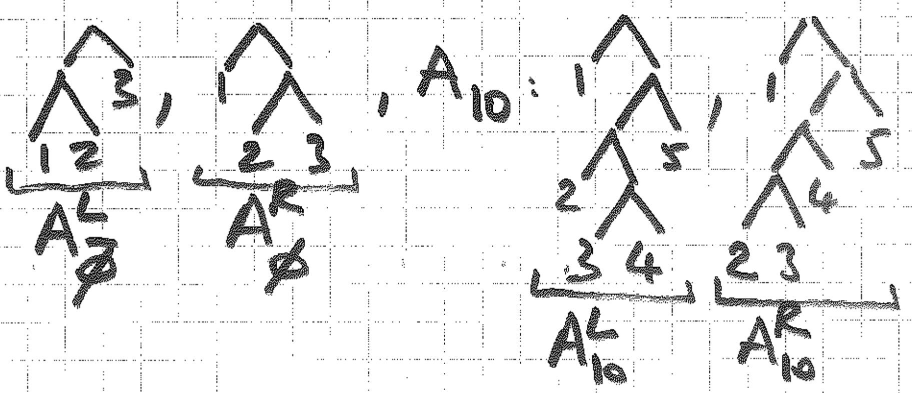
Lemma. For every law I and every w in $(A \cup \bar{A})^*$ if I_w is not empty, there exist terms I^L_w, I^R_w s.t. I_w is the family of all instances of (I^L_w, I^R_w).

Proof. Induction on $|w|$.

* $|w| = 1$, i.e., $w = \alpha^e$
  - $w = \emptyset$: OK with (I^L, I^R)
  - $w = \bar{\emptyset}$: OK with (I^R, I^L)
  - $w = 0\alpha$: $I^L_{0\alpha} = \wedge\, I^L_\alpha$ $I^R_{\alpha} = \wedge\, I^R_\alpha$ ...

Example.

$A_{\emptyset}$:   , A_{10} :



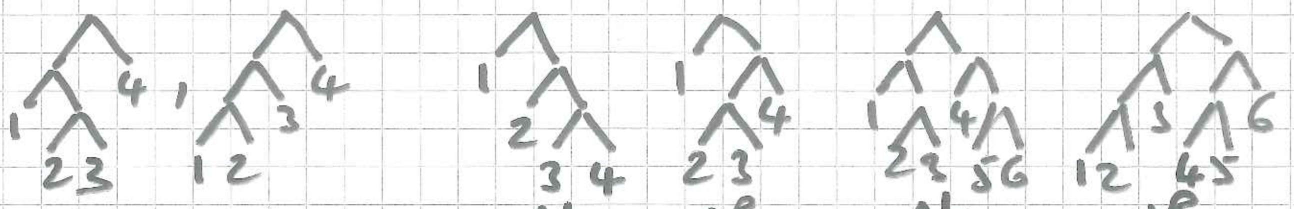
* $w = uv$

$$I^L_u \sigma \xmapsto{I_u} I^R_u \sigma$$
$$I^L_v \tau \xmapsto{I_v} I^R_v \tau$$

Claim. If $\exists \sigma, \tau \, (I^R_u \sigma = I^L_v \tau)$, then $\exists$ least one σ_0, τ_0 "unifier"

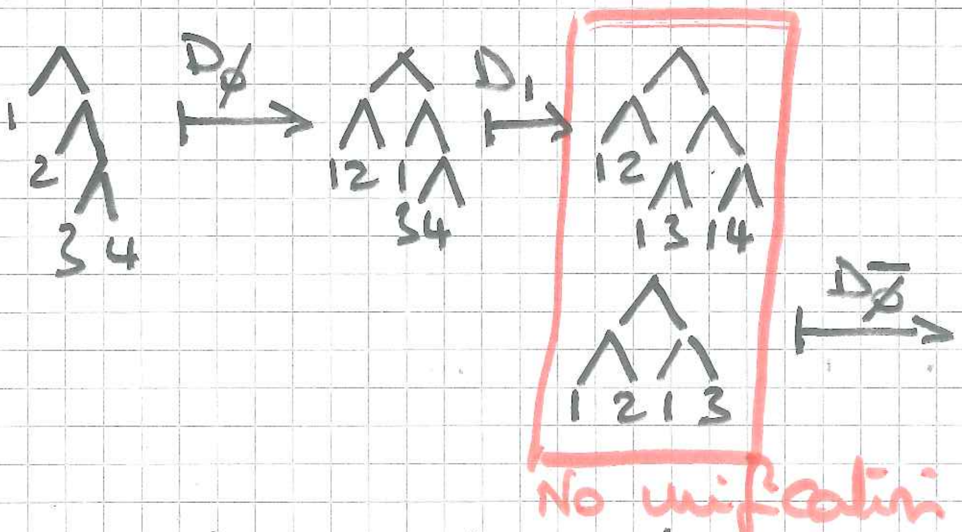
Then $I^L_w = I^L_u \sigma_0$ and $I^R_w = I^R_v \tau_0$.

Example. Case of A: A^L and A^R are injective terms (no variable repetition), whence A^L_w and A^R_w exist and are injective for all w: unification $= \cup$ skeletons.



A^L_0 A^R_0 A^L_1 A^R_1 $A^L_{0|1}$ $A^R_{0|1}$

But



$D_{\emptyset|1|\bar\emptyset}$ is empty.

3. Making $\mathcal{G}_I$ into a group

Fact. $\mathcal{G}_I$ is an inverse monoid

$$\forall g\, \exists \bar g\, (g \bar g g = g \ \& \ \bar g g \bar g = \bar g)$$

but not a group:

$$g\bar g = \mathrm{id}_{\mathrm{Dom}\, g} \neq \mathrm{id}_{\mathcal{G}}$$

Lemma. For g, g' in $\mathcal{G}_I$, declare $g \simeq g'$ iff

$$\exists T\, (T.g\downarrow \text{ and } T.g'\downarrow)$$
$$\forall T\, ((T.g\downarrow \text{ and } T.g'\downarrow) \Rightarrow T.g = T.g')$$

Assume

(*) $\forall g_1 \ldots g_n\, \exists T\, (T.g_1\downarrow \text{ and} \ldots \text{and } T.g_n\downarrow)$

(**) $\exists T\, (T.g = T.g') \Rightarrow g \simeq g'$

Then $\simeq$ is a congruence and $\mathcal{G}_I/\simeq$ is a group.

Proof. (Easy).

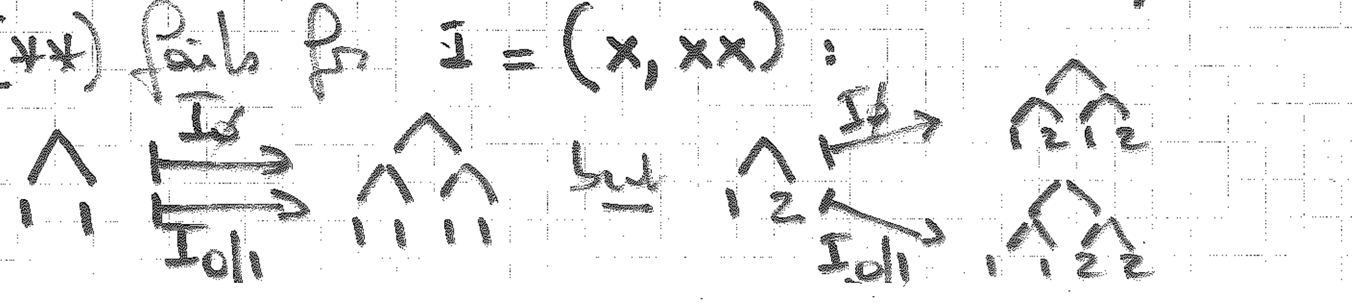
Example. (A)

(*) $T.A_W\downarrow$ iff $\mathrm{skel}(T) \supseteq \mathrm{skel}(A^L_W)$.

(**) A^L_W and A^R_W are injective terms.

But (*) fails for (D): $T.D_{\bar\emptyset}\downarrow$ & $T.D_{\bar 1\emptyset}\downarrow$ imp.

(**) fails for $I = (x, xx)$:



So: $\boxed{\mathcal{G}_A/\approx \text{ is a group.}}$

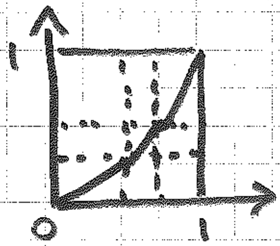
Prop (R. Thompson, 1963) $\mathcal{G}_A/\approx \; = F$.

Proof.

$$\approx\text{-class of } w \longmapsto \left(\bigcap_{w' \approx w} \underline{\text{skel}}(A^L_{w'}) , \bigcap_{w' \approx w} \text{skel}(A^R_{w'}) \right)$$

■

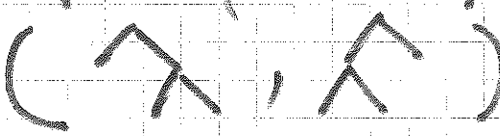
Recall: $F := \{ f \in \underline{\text{Homeo}}^+([0,1]) \mid$ piecewise affine, dyadic slopes, dyadic changes $\}$



$\longleftrightarrow$ pair of dyadic decompositions of $[0,1]$

or

$\longleftrightarrow$ pair of binary trees



Exercise: $\mathcal{G}_{A,C}/\approx \; = V$

commutativity $xy = yx$