~~So we have~~

# III Cluster category / cluster tilting objects

recall mutation of tilting module
is sometimes not possible:
some almost complete tilt. mod have 1 complement
$\rightsquigarrow$ how to fix that?
use instead cluster tilt$^g$ objects in cluster category
(another way = use support $\tau$-tilting objects)

derived cat. $D^b \operatorname{mod} kQ$
usual definition: $\operatorname{Ch} kQ$ cat. of chain complexes

$$\to M_i \xrightarrow{d_i} M_{i+1} \xrightarrow{d_{i+1}} \cdots \qquad i \in \mathbb{Z}$$

bounded: $M_i = 0$ if $|i| \gg 0$

quasi iso = morphism of ch. complex inducing isomorph in homology

example $Q = 1 \to 2$

$$\begin{cases} 0 \to 11 \xrightarrow{d} 10 \to 0 \text{ ch. cplx} \\ \quad f\uparrow \qquad \uparrow \\ 0 \to 01 \xrightarrow{d} 0 \to 0 \text{ ch cplx} \\ \quad i=0 \qquad i=1 \end{cases}$$

not invertible
force to be invertible by adding formal inverses
new cat: $D^b \operatorname{mod} kQ$

in which $\underset{i=0}{11} \xrightarrow{d} 10 \;\simeq\; \underset{i=0}{01}$

* on $D^b \operatorname{mod} kQ$

there is a shift functor $\Sigma : D^b \operatorname{mod} kQ \to$ itself

it maps a chain complex $\to M_i \xrightarrow{d_i} M_{i+1} \to \cdots$ (with $M_i$ in degree $i$)

to the shifted chain complex $M_{i+1} \xrightarrow{-d_i} M_{i+2}$

defines an auto equivalence of $D^b \operatorname{mod} kQ$

* on $D^b \operatorname{mod} kQ$, the A-R translation functor $\tau$ becomes an autoequivalence

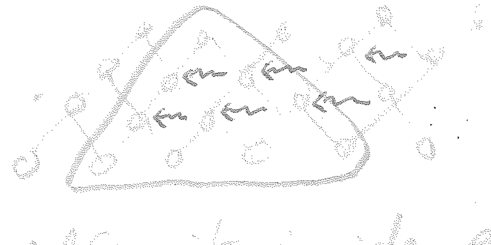
in fact

Let us restrict to ~~an~~ on $Q = \vec{A}_m = \circ \to \circ \to \circ \to \circ \to \circ$

Auslander-Reiten theory gives a presentation of $D^b \operatorname{mod} kQ$

* indecomposables of $D^b \operatorname{mod} kQ \longleftrightarrow$ pairs $(M, k)$, $M$ indec $\operatorname{mod} kQ$, $k \in \mathbb{Z}$

* A-R quiver (irreducible maps, action of $\tau$) looks like an horizontal strip of height $n$



module cat. $\operatorname{mod} kQ$ sits inside as full subcat.

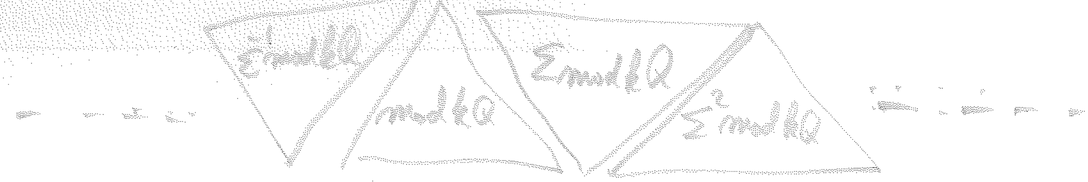
AR functor $\tau$ = translation $\leftarrow$ to the left

shift functor $\Sigma$ = glide reflection to the right

△ ~ ▽

so the AR quiver of $D^b \operatorname{mod} kQ$ decomposes as






derived categories allow to write $\operatorname{Ext}^i$ just ~~using~~ $\operatorname{Hom}_{D^b}$ and $\Sigma$ (very general)

$$\operatorname{Ext}^1(A,B) = \operatorname{Hom}_{D^b}(A, \Sigma B)$$

Also have another formula:

$$\operatorname{Hom}_{D^b}(A, \tau\Sigma B) \simeq \operatorname{Hom}_{D^b}(B, A)^*$$

coming from the symmetry of arrows seen before

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Now : cluster category

main idea: force $\operatorname{Ext}^1(A,B) \simeq \operatorname{Ext}^1(B,A)^*$

technical tool = ~~quotient~~ orbit category

Def: $\boxed{\mathcal{C}_Q = \text{orbit category } D^b\operatorname{mod} kQ \text{ for auto equiv } \tau\Sigma^{-1}}$

<u>Property</u> ~~[illegible]~~ $\tau = \Sigma$ on $\mathcal{C}_Q$

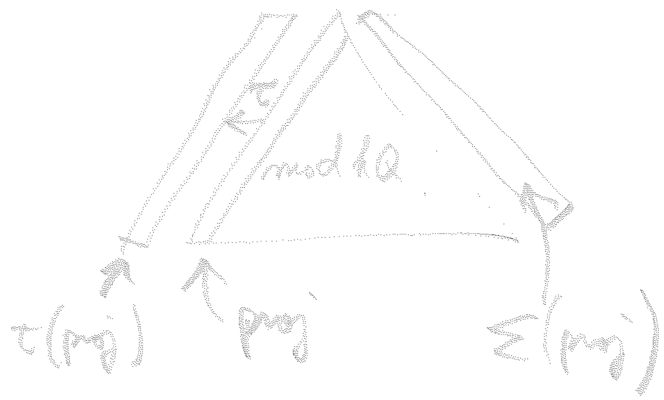
hence $\operatorname{Ext}^1_{\mathcal{C}}(A,B) = \operatorname{Hom}_{\mathcal{C}}(A, \Sigma B) = \operatorname{Hom}_{\mathcal{C}}(\tau A, \Sigma\tau B)$

$\simeq \operatorname{Hom}_{\mathcal{C}}(B, \tau A)^* = \operatorname{Hom}_{\mathcal{C}}(B, \Sigma A)^* = \operatorname{Ext}^1(B,A)^*$

as wanted.

# A-R-quiver of cluster category $\mathcal{C}_Q$






in which we <u>identify</u> $\tau(P_i)$ and $\Sigma(P_i)$

result is a Möbius band

<u>Example</u> $1 \to 2 \to 3$

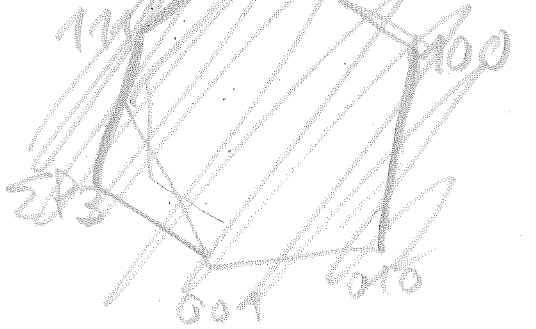
$$\begin{tikzcd}[column sep=small]
 & & \tau P_1 \arrow[dr] & & 111 \arrow[dr] & & \Sigma P_3 \arrow[dr] & & \\
 & \tau P_2 \arrow[ur] \arrow[dr] & & 011 \arrow[ur] \arrow[dr] & & 110 \arrow[ur] \arrow[dr] & & \Sigma P_2 \arrow[dr] & \\
\tau P_3 \arrow[ur] & & 001 \arrow[ur] & & 010 \arrow[ur] & & 100 \arrow[ur] & & \Sigma P_1
\end{tikzcd}$$

so there are $6+3$ indecomp. objects in $\mathcal{C}$

<u>Claim</u>: natural bijection for $\mathcal{C}_Q$ ($Q = \vec{A}_n$)

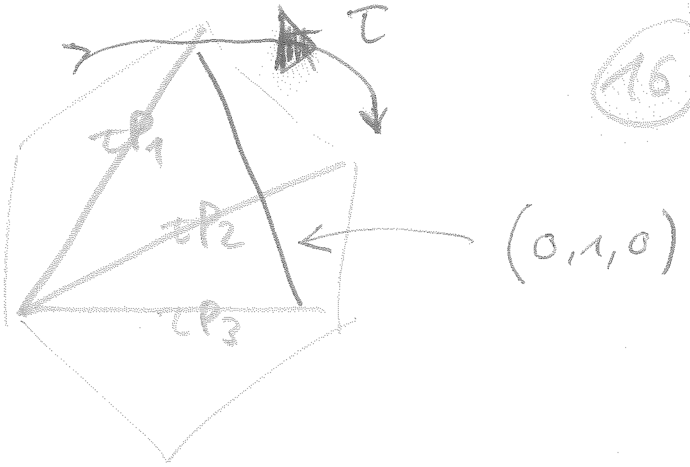
indec objects in $\mathcal{C}_Q$ $\longleftrightarrow$ inner diagonals in a regular polygon with $n+3$ vertices

+ action $\tau$




+ rotation

$\tau(P_i) = (0,0,\overset{i}{-1},0,0)$




other diagonals $d$ = ~~[illegible]~~ sum $\left(-v,\ v \in \{\tau P_{1,2,3}\},\ v \text{ crosses } d\right)$

<u>Claim</u>: by this bijection
$$\mathrm{Ext}^1(A,B) \neq 0 \iff d_A \text{ crosses } d_B$$

---

cluster tilting objects in $\mathcal{C}_Q$ (n vertices)

<u>Def</u>
$$\begin{cases} T = T_1 \oplus \cdots \oplus T_n \\ \quad T_i \text{ pairwise distinct indecom. objects} \\ \mathrm{Ext}^1(T,T) = 0 \quad (\text{also called } \underline{\text{rigidity}}) \end{cases}$$

<u>Def</u> almost complete c.t.o | <u>Rem</u> $\boxed{\text{contains tilting object of mod } kQ}$

<u>Thm</u> (B.M.R.R.T.)
any almost complete clust. tilt. ob. has 2 compl.

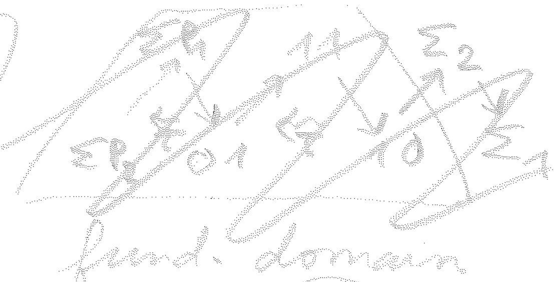
⇝ one can always define the mutation of a cluster-tilting object $T = T_1 \oplus \cdots \oplus T_n$ at ~~some~~ <u>any</u> index $i$

10 ⇝ one can define mutation graph (whose

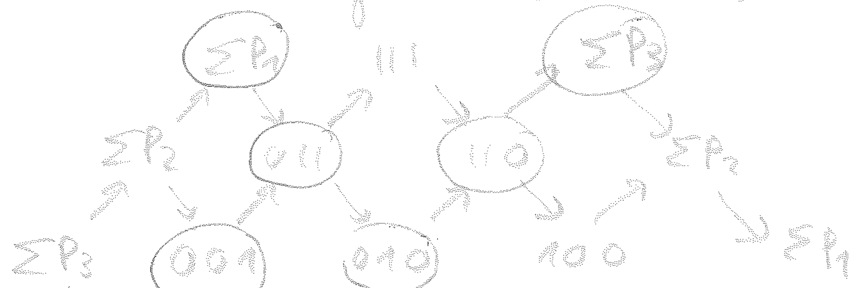
vertices = cluster tilting objects

edges = mutations

Example: $1 \to 2$




$1 \to 2 \to 3$




some clust. tilt objects

Claim for $Q = \vec{A}_n$

cluster tilting objects $\simeq$ sets of $n$ indec without $\mathrm{Ext}^1$ $\longleftrightarrow$ triangulations = sets of $n$ diagonals with no crossing

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so the mutation graph = flip graph of triangulations = vertices and edges of associahedron

# IV Cluster algebra, cluster character

$$Q = \vec{A}_n$$

~~Claim~~ Let

~~Ex~~ $x_1, \ldots, x_n$ $n$ indeterminates generate $\mathbb{Q}(x_1, \ldots, x_n)$

Thm (Caldero-C)

~~Claim~~ : $\exists !$ map $\mathrm{Indec}(\mathcal{C}_Q) \xrightarrow{cc} \mathbb{Q}(x_1, \ldots, x_n)$

sending $\Sigma P_i \longmapsto x_i$

and such that* for every

almost-split seq. $0 \to A \to B \to C$

$$cc(A)\,cc(C) = cc(B) + 1$$

(moreover ~~and~~ cc extends to all objects * $cc(A \oplus B) = cc(A) \times cc(B)$)

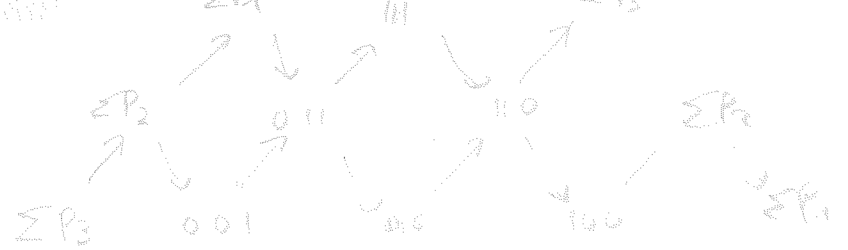
× something "like a determinant"

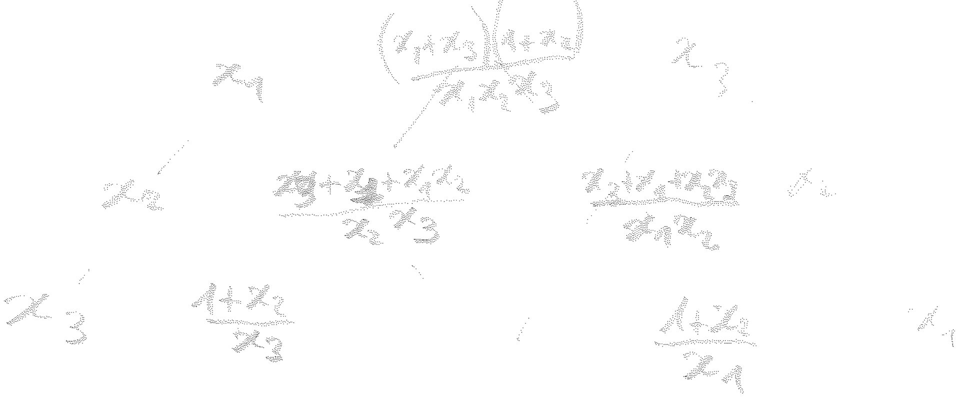
if $A, B$ would be matrices

$$\det(A \oplus B) = \det(A) \times \det(B)$$

× uniqueness follows from required properties

× existence uses an explicit formula containing Euler charact. of quiver Grassmannian example $\overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ}$




~~The lamont polynomials defin~~

What one gets are "cluster variables" in the cluster algebra attached to the quiver Q: 3-sided correspondence:

| | | |
|---|---|---|
| Indecomp. obj. of $\mathcal{C}_Q$ | diagonals in polygon | cluster variables |
| cluster tilting obj | triangulations | clusters |
| mutation | flip | mutation |

× the category $\mathcal{C}_Q$
is a global object describing the combinatorics
of cluster algebra (has global symmetries, etc)

* mutation of cluster variables
is explained by ( CC map behaviour
+ mutation of clust. tilt objects

$CC(T')\,CC(T'') = \text{monomial} + \text{monomial}$
correspond to two s. exact seq. $T' \hookrightarrow A \twoheadrightarrow T''$ & $T'' \hookrightarrow A \twoheadrightarrow T'$

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## V representations of Tamari lattices

we turn to something else

Tamari lattice seen as an algebra or (large) quiver

to every finite poset $P \rightsquigarrow$ incidence algebra
$\mathrm{Inc}_k(P)$ : basis $e_{x,y}$ for $\{(x,y) \in P^2 \mid x \leq y\}$

field

product $e_{x,y}\, e_{z,t} = \underbrace{\delta_{y=z}}_{\text{kronecker delta}}\, e_{x,t}$

Prop $\mathrm{Inc}_k(P)$ is a finite dim. assoc alg / $k$
unit $= \sum_{x \in P} e_{xx}$

~~Pfhly~~

Prop $\boxed{\mathrm{Inc}_k(P) \text{ has finite homological dim}}$ (21)

meaning : $\exists N$ s.t every ~~k-
module~~ $\mathrm{Inc}_k(P)$-mod has a ~~finite~~ proj resolution of length at most $N$

$$0 \to P_N \to P_{N-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

Proof one can take $N = \# P$

indeed every module has a support $S \subseteq P$

choose one minimal elt $i$ of $S$

then $0 \to M' \to P_i^{\oplus m_i} \to M \to 0$

with support $M' \subseteq \mathrm{Supp}\, M - \{i\}$

~~Exercice~~ (better bound: length of maximal chain)

Remark categ mod $\mathrm{Inc}_k P$

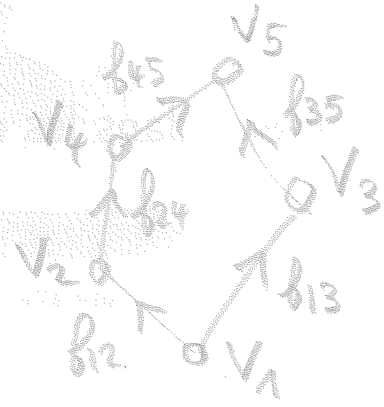
$\simeq$ categ of reps of the Hasse diagram of $P$ with commutation relations

Hasse diagram $\begin{cases} \text{vertices} = \text{elt of } P \\ \text{edges} = \text{cover relations } (x \to y) \end{cases}$

if $x < y$ and $x \le z \le y$
$\Rightarrow z = x$ or $z = y$

Example

reps of $\mathrm{Tamari}_2$




such that

$b_{13}b_{35} = b_{12}b_{24}b_{45}$

Why study modules over Tamari lattice?

start is an observation

$M =$ matrix $1_{x\le y}$ $\qquad \begin{pmatrix}1 & & 0\\ & \ddots & \\ 1\text{ or }0 & & 1\end{pmatrix}$

$C = -M({}^tM)^{-1}$

~~Thm~~ $\boxed{C^{2n+2} = Id \text{ for the Tamari lattice } T_n}$

More precisely:

let $\lambda(n) = (-1)^{\binom{n}{2}}\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}$ $\quad n\geq 1$

let $b_n = \frac{1}{n}\sum_{d|n}\mu(d)\lambda(n/d) \in \mathbb{Z}$

Möbius function

thm char polynomial of C is

$$\frac{\left(x^{2n+2}-1\right)^{C_n}}{\left(\prod_{d|2n+2}\left(x^d-(-1)^{d(n+1)}\right)^{b_d}\right)^{(-1)^{n+1}}}$$

$b_n = 1, -1, -1, 1, 1, -1, -3, 4, 8, -13$

Tam 4

14 elts

$c^{10} = 1$

23

$\chi = \Phi_2^2 \Phi_5 \Phi_{10}^2$

formula gives $\dfrac{(x^{10}-1)^{14}}{(x+1)^{-1}(x^2-1)^{1}(x^5+1)^{-1}(x^{10}-1)^{13}}$

$\checkmark$ ok