III Cluster category / cluster tilting objects

Recall mutation of tilting module is sometimes not possible. Some almost complete tilt. mod have 1 complement.

\[ \text{How to fix that?} \]

Use instead cluster tilt? objects in cluster category (another way = use support \( T \)-tilting objects)

derived cat. \( D^b \text{mod } kQ \)

Usual definition: \( CH \& Q \) cat. of chain complexes

\[ M_i \xrightarrow{d_i} M_{i+1} \to \cdots \]

Bounded: \( M_i = 0 \) if \( |i| \gg 0 \)

Quasi iso = morphism of ch. complex inducing isomorph in homology

Example \( A = 1 \to 2 \)

\[ \begin{array}{c}
0 \to 10 \to \text{ch. cycle} \\
\uparrow \\
0 \to 0 \to \text{ch. cycle}
\end{array} \]

Not invertible

Force to be invertible by adding formal inverses

New cat: \( D^b \text{mod } kQ \)

In which \( 10 \cong 0 \)
On $\text{D}^b(\text{mod } kQ)$ there is a shift functor $\Sigma : \text{D}^b(\text{mod } kQ) \to \text{ itself}$.

It maps a chain complex $\rightarrow M_i \xrightarrow{d_i} M_{i+1} \rightarrow \cdots$

to the shifted chain complex $\rightarrow M_{i+1} \xrightarrow{d_i} M_{i+2} \rightarrow \cdots$

defines an auto-equivalence of $\text{D}^b(\text{mod } kQ)$.

On $\text{D}^b(\text{mod } kQ)$, the A.R. translation functor $\tau$ becomes an auto-equivalence.

In fact, let us restrict to an ad $\text{ad } A = \text{Am} = 0 \rightarrow \ldots \rightarrow \infty$

Keller's theory gives a presentation of $\text{D}^b(\text{mod } kQ)$.

$X$ indecomposable of $\text{D}^b(\text{mod } kQ) \rightarrow$ points $(M,k)$

$\text{Ind}_{\text{mod } kQ}$

$X$. A.R. quiver irreducible maps, action of $\tau$?

Look like an horizontal strip of height

module cat $\text{mod } kQ$ sits inside as full subcat.

A.R. functor $\tau$ = translation far to the left.

Shift fun. $\sigma$ = glide reflection to the right.
Derived categories allow to write \( \text{Ext}^\bullet \) just using \( \text{Hom}_D \) and \( \Sigma \) (very general)

\[
\text{Ext}^\bullet(A, B) = \text{Hom}_D(A, \Sigma B)
\]

Also have another formula:

\[
\text{Hom}_D(A, \Sigma B) = \text{Hom}_E(B, A)
\]

Coming from the symmetry of arrows seen before.

Now: cluster category

Main idea: force \( \text{Ext}^\bullet(A, B) = \text{Ext}^\bullet(B, A)^* \)

Technical tool = quotient category

Def: \( E\Sigma Q = \text{orbit category } D\Sigma \text{mod} \Sigma Q \) for auto equiv \( \tau \Sigma^{-1} \)

Property: \( \tau = \Sigma \) on \( E\Sigma Q \)

Hence

\[
\text{Ext}^\bullet(A, B) = \text{Hom}_E(A, \Sigma B) = \text{Hom}_E(\Sigma A, \Sigma B) = \text{Hom}_E(B, \Sigma A) = \text{Ext}^\bullet(B, A)^*
\]

As wanted.
AR-genuine of cltwo categories Cq

\[ \tau \text{ mod } \lambda \]

\[ \pi_1 \xrightarrow{\text{proj}} \pi_2 \xrightarrow{\Sigma} \pi_3 \]

in which we identify \( \tau(P_i) \) and \( \Sigma(P_i) \).

Result is a Möbius band.

Example \( 1 \rightarrow 2 \rightarrow 3 \)

\[ \tau P_1 \quad \pi_1 \quad \Sigma P_2 \]

\[ \tau P_2 \quad 001 \quad 010 \quad 100 \quad \Sigma P_3 \]

So there are 6 + 3 indecomposable objects in \( C \).

Claim: natural bijection for \( C_q (q = A_m) \)

indec objets \( \leftarrow \) diagonals in a regular polygon in \( C_q \)

+ action \( T \)

+ rotation
\( \tau(\mathbf{i}) = (0, 0, -1, 0, 0) \)

Other diagonals = \( \ell \text{th sum} (-v, v \in \{0, 2, 3, 5\}, \forall \text{ crossed} \) \( d \)

Claim: by this bijection
\[ \text{Ext}^n(A[B]) \neq 0 \iff d_A \text{ crosses } d_B \]

Cluster tilting object in \( \mathcal{E}_{\mathcal{Q}} \) (\( n \) vertices)

Def: \( T = T_1 \oplus \ldots \oplus T_n \)
\[ T_i \text{ pairwise distinct indecomposable object} \]
\[ \text{Ext}^n(T_i, T) = 0 \] (called rigidity)

Def: almost complete c.t.o | Rem: contains tilting orbit \( \text{mod } kQ \)

Thm (B.M.R.R.T.)
Any almost complete cluster tilting has 2 complements

One can always define the
\[ T = T_{i_0} \oplus T_m \text{ at some } \text{ index } i \]
one can define mutation graph whose
vertices = cluster tilting objects
edges = mutations

Example:

\[ 1 \to 2 \to 3 \]

some cluster tilting objects

claim for \( Q = \mathbb{F}_m \)
cluster tilting objects
\( \cong \) some sets of indece without Ext^2

so the mutation graph = flip graph of triangulations
vertices and edges of associahedron
Let \( Q = \overline{A^m} \).

Then \((\text{cdder} - c)\)

Claim: \( \exists \) ! map \( \text{Indec}(\mathscr{E}_0) \rightarrow Q(x_1, \ldots, x_m) \) sending \( \sum \mathbf{P}_i \rightarrow x_i \)
and such that for every almost split seq. \( 0 \rightarrow A \rightarrow B \rightarrow C \)

\[ \text{cc}(A) \cdot \text{cc}(C) = \text{cc}(B) + 1 \]

(moreover \( x \cdot \text{cc}(A \oplus B) = \text{cc}(A) \times \text{cc}(B) \))

Something "like a determinant".

If \( A, B \) would be matrices

\[ \text{det}(A \oplus B) = \text{det}(A) \times \text{det}(B) \]

Uniqueness follows from required properties.

Existence uses an explicit formula containing Euler character of quiver Grassmannian.
The polynomials define what one gets are "cluster variables" in the cluster algebra attached to the quiver $Q$: 3-sided correspondence.

Cluster variables in polygon

Cluster tilting Obj: triangulations, clusters

Mutation, flip, mutation
The category $E_q$ is a global object describing the combinatorics of cluster algebra (has global symmetries, etc.)

* mutation of cluster variables
  is explained by (CC map behaviour
  + mutation of cluster tilt object

\[ CC(t^+) \circ CC(t^-) = \text{monomial} + \text{monomial} \]

correspond to two subsets seq. $t_{-} \rightarrow A \rightarrow t_{+}$ & $t_{-} \rightarrow A \rightarrow t_{+}$

V. Representations of Tamari lattices

We turn to something else (large)

Tamari lattice seen as an algebra or quiver

To every finite poset $P \rightarrow$ incidence algebra

\[ \text{Inc}(P) : \text{basis } e_{x,y} \text{ for } \{ (x, y) \in P^2 \mid x < y \} \]

product $e_{x,y} e_{z,t} = \delta_{y=z} e_{x, t}$

field

\[ \text{Proj } \text{Inc}(P) \text{ is a finite dimensional associative algebra } k \]

\[ \text{Unit } = \sum_{x \in P} e_{xx} \]
Prop. \( \text{Incf}(P) \) has finite homological dim

meaning: \( \exists N \) s.t. every \( \text{Incf}(P) \)-mod has a finite proj resolution of length at most \( N \)

\[
0 \to P_N \to P_{N-1} \to \cdots \to P_1 \to P_0 \to M \to 0
\]

Proof: one can take \( N = \# P \)

indeed every module has a support \( S \subseteq P \)
choose one minimal elt of \( S \)

then \( 0 \to M' \to P_{\sigma_{\text{min}}} \to M \to 0 \)

with support \( M' \subseteq S \cup \{ \text{i} \} \)

Excerpts (better bound: length of maximal chain)

Remark: \( \text{categ. mod Incf}(P) \)

= categ. of rings of the Hasse diagram of \( P \)

with commutation relations

Hasse diagram vertices = elt of \( P \)
edges = cover relations \( (x \to y) \)
if \( x \leq y \) and \( x \neq y \)

\[ x \neq y \]
Why study modules over Tamari lattices?

Start is an observation

\[ M = \text{matrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & 1 \end{pmatrix} \]

\[ C = -M(M)^{-1} \]

Theorem: \[ C^{2m+2} = \text{Id} \] for the Tamari lattice \( T_m \)

More precisely:

let \( a(n) = (-1)^{\lfloor n/2 \rfloor} \quad n \geq 1 \)

let \( b_n = \frac{\sum_{d|m} \mu(d) a\left(\frac{m}{d}\right)}{n \, \text{rad}(m)} \quad \in \mathbb{Z} \)

Then the characteristic polynomial of \( C \) is

\[ (x^{2m+2} - 1)^m \]

\[ + \left( \frac{\Gamma(\frac{1}{2}) \Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} \right) (-1)^{m+1} \]
\[ b_n = 1, -1, -1, 1, 1, -1, -3, 4, 4, 13 \]

\[ X = \phi_2 \phi_5 \phi_{10} \]

Formula gives

\[ \frac{(x^{20} - 1)^4}{(x+1)^{-1} (x^2 - 1) (x^5+1)^{-1} (x^{10} - 1)^3} \]

\[ \checkmark \]