+Ref 1012.4949 (Reiten) (1)

CIRM

→ quiver repr. → tilting
→ cluster cat (+ cluster char)
→ modules over Tamari lattices

I Quiver repr

$Q = (Q_0, Q_1, s, t)$ quiver $\quad Q_0$ = set of vertices, Q_1 = set of edges

$Q_1 \xrightarrow{s} Q_0$ (source) $\quad Q_1 \xrightarrow{t} Q_0$ (target) $\quad$ (means $s(f) \xrightarrow{f} t(f)$)

quiver = oriented graph, can have loops, can have mult. edges

we will concentrate on quivers that are orientations of the path graph

o—o—o—o ... o—o

quiver representations over a field k

Def a representation V of Q ~~in the cat~~ over k is a collection of fin. dim. k-vector spaces V_i $i \in Q_0$ and linear maps $V_i \xrightarrow{V_f} V_j$ for $f \in Q_1$, $i = s(f)$, $j = t(f)$

morphisms of repr. : $V_i \xrightarrow{g_i} W_i$ such

$$\begin{tikzcd} V_i \arrow[r, "V_f"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\text{comm.}"] & V_j \arrow[d, "g_j"] \\ W_i \arrow[r, "W_f"'] & W_j \end{tikzcd} \qquad \forall f \qquad \text{(also isom)}$$

(modules over path alg)

This defines a category $\mathrm{mod}\, kQ$

→ ~~add~~ k-linear : { Hom(V,W) k-vector space; comp. is k-bilinear

→ additive : one can define $(V \oplus W)_i = V_i \oplus W_i$

→ abelian : Kernels and cokernels

$(\mathrm{Ker}\, V \xrightarrow{g} W)_i = \mathrm{Ker}\, V_i \xrightarrow{g_i} W_i$

<u>Def</u> a repr. of Q is

* simple if it has no nontrivial sub~~module~~ repr.
* indecomposable if it cannot be written as $W' \oplus W''$ with $W' \neq 0$ and $W'' \neq 0$

<u>Prop</u> $\mathrm{mod}\, kQ$ has the Krull-Schmidt property:

if $W \simeq W_1 \oplus W_2 \oplus \cdots \oplus W_k \simeq W'_1 \oplus W'_2 \oplus \cdots W'_\ell$ with W_i and W'_j indecomp. then $k = \ell$ and there is a permutation σ such that $W_i \simeq W'_{\sigma i}$

<u>Example</u> $Q = \overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ}$

repr. $k \xrightarrow{0} 0 \xrightarrow{0} 0 \qquad 0 \to k \to 0 \qquad 0 \to 0 \to k$

$k \xrightarrow{1} k \to 0 \qquad 0 \to k \xrightarrow{1} k$

$k \xrightarrow{1} k \xrightarrow{1} k$

CIRM

Claim:

- these 6 repr are indecomp. (easy)
- every indecomp is isomorphic to one of them
- they are pairwise not isomorphic (easy)

Beware: for general Q, there can be infinitely many ~~iso~~ isom. classes of indec. repr.

Def: dimension vector of a repr. V
$= (\dim V_i)_{i \in Q_0} =: \underline{\dim} V$

for example $k \xrightarrow{1} k \to 0$ has $\underline{\dim} V = (1,1,0)$

Theorem (Pierre Gabriel) special case 1972

Let Q be any orientation of o—o—o—o…o—o (n vertices)
Then Q has $\binom{n+1}{2}$ isom. classes of indecomp. repr
The dim. vectors of indec are all distinct, given by $(0,0,0,\underset{\uparrow}{1},1,\dots,\underset{\uparrow}{1},0,0\,0)$ for

$1 \leq i \leq j \leq n$

explicit description of indecomp. repr:

$$0 \to 0 \to \underbrace{k \to k \to \dots \to k}_{i \quad\quad j} \to 0 \to \dots \to 0$$

CIRM - Centre International de Rencontres Mathématiques
Campus de Luminy - Case 916 - F 13288 Marseille cedex 9 - T + 33 (0)4 91 83 30 00 - Fax + 33 (0)4 91 83 30 05
www.cirm-math.fr - twitter.com/@_CIRM - google + CIRM - youtube.com/CIRMchannel - www.chairejeanmorlet.com

At this point, we understand all objects of mod kQ : direct $\oplus$ of indec.
What about morphisms?

$*$ $\mathrm{Hom}(A \oplus B, C) \simeq \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C)$
$\quad \mathrm{Hom}(A, B \oplus C) \simeq \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C)$

direct sum is prod. and coprod.

so $\rightarrow$ enough to understand morphisms between indecomposables

$1 \to 2 \to 3$

$$
\begin{tikzcd}
& & 111 \arrow[dr] & & \\
& 011 \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\text{comm}"] & & 110 \arrow[dr] & \\
001 \arrow[ur] \arrow[rr, bend left, "0"] & & 010 \arrow[ur] \arrow[rr, bend left, "0"] & & 100
\end{tikzcd}
$$

this is a presentation of the categ. mod kQ.
(or rather of its restriction to indec. repr)

arrows = irreducible morphisms f
~~meaning : if $f = gh$ then~~ $\begin{cases} \cancel{g = \lambda \mathrm{Id}} \\ \cancel{\text{or}} \\ \cancel{h = \mu \mathrm{Id}} \end{cases}$

$*$ f is not an isom

$*$ $\forall$
$$
\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "\alpha"'] & \text{com} & W \\
& U \arrow[ur, "\beta"'] &
\end{tikzcd}
$$
then α is split monomorph $\rightarrowtail$
or
β is split epimorph $\twoheadrightarrow$

This presentation comes from the Auslander - Reiten theory
(Lecture Notes by Lidia Angeleri Hügel)

Recall Def using $\oplus$

CIRM - Centre International de Rencontres Mathématiques
Campus de Luminy - Case 916 - F 13288 Marseille cedex 9 - T + 33 (0)4 91 83 30 00 - Fax + 33 (0)4 91 83 30 05

$1 \to 2$ $\quad P_2^2 \nearrow^{I2 = P_2}$

* proj: follow the arrows

uses a notion of almost-split sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{(exact)}$$

f not a split mono and

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & X \\ & B \arrow[u, dashed] \end{tikzcd}$$

$A \to X$ not split mono factors

g not a split epi and

$$\begin{tikzcd} X \arrow[r] \arrow[dr, dashed] & C \\ & B \arrow[u, "g"'] \end{tikzcd}$$

$X \to C$ not split epi factors

- given A not injective, exists and unique
- given C not projective*, ———

$\rightsquigarrow$ partial functor τ : not proj $\longrightarrow$ not inj
invertible $\qquad C \longmapsto A$

~~Property 2 Hom~~

<u>Key point</u> : almost split sequences gives us the <u>relations</u> in the cat. mod kQ

$1 \to 2 \to 3$

proj $\qquad\qquad$ Inj

$$\begin{tikzcd}
 & & 111 \arrow[dr] & & \\
 & 011 \arrow[ur] \arrow[dr] & & 110 \arrow[ll, squiggly, "\tau"'] \arrow[dr] & \\
001 \arrow[ur] & & 010 \arrow[ll, squiggly, "\tau"'] \arrow[ur] & & 100 \arrow[ll, squiggly, "\tau"']
\end{tikzcd}$$

this called the AR-quiver:

vertices = isom classes of indec

arrows = irreducible morph.

then $*$ arrows are stable by τ: as many

$*$ arrows $[M,N] \simeq [\tau M, \tau N]$ arrows
when makes sense

$*$ as many $[N, M]$ arrows as $[\tau M, N]$ arrows when makes sense

$$
\begin{tikzcd}
 & 111 \arrow[dr] & \\
011 \arrow[ur] \arrow[dr] & & 110 \\
 & 010 \arrow[ur] &
\end{tikzcd}
$$

<u>Summary</u>: full description of the category mod kQ for Q an orientation of $\circ\circ\circ\circ\circ$

Now let us use that.

II <u>tilting modules</u>

Ext groups: A, B objects $\mathrm{Ext}^i(A,B)$ abelian groups $i \geq 0$

$\mathrm{Ext}^0(A,B) = \mathrm{Hom}(A,B)$

higher ext are defined using any project resolution of A: $0 \to P^1 \to P^0 \to A \to 0$

hereditary cat $\leftarrow$ (note that $\exists$ such a short resolution) for mod kQ

$\rightsquigarrow$ complex $0 \to \mathrm{Hom}(P^1,B) \to \mathrm{Hom}(P^0,B) \to 0$

$\rightsquigarrow$ homology $\mathrm{Ext}^1(A,B) \oplus \mathrm{Hom}(A,B)$

concretely $Ext^1(A,B) \neq 0$
means that there exists non-split short exact sequences $0 \to B \to M \to A \to 0$

M is called an extension of A by B

<u>Def</u>: a partial tilting module T
is a direct sum $T = T_1 \oplus \dots \oplus T_k$
of indecomp repr. such that
$Ext^1(T,T) = 0$ $\left(\begin{array}{l}\text{equivalent to } \forall i,j \\ Ext^1(T_i, T_j) = 0\end{array}\right)$

<u>Def</u> a <u>tilting module</u> is a partial till. module such that
* all T_i are pairwise non isom^ic
* if $T \oplus \underset{\text{indec}}{T'}$ is partial tilting then $T' \simeq T_i$ for some i

<u>examples</u>

$$\overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ}$$

$$\begin{tikzcd}
 & & 111 \arrow[dr] & & \\
 & 011 \arrow[ur] \arrow[dr] & & 110 \arrow[ll, dashed] \arrow[dr] & \\
001 \arrow[ur] & & 010 \arrow[ll, squiggly] \arrow[ur] & & 100 \arrow[ll, squiggly]
\end{tikzcd}$$

$\boxed{\text{Ext of proj} = 0}$

* $001 \oplus 111$ is partial tilting
* $001 \oplus 111 \oplus 100$ is tilting

* all proj (left)
* all inj (right)

are both tilting

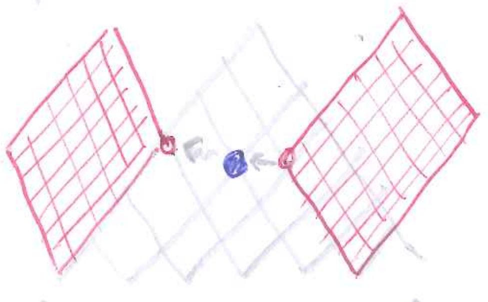
<u>Rem</u> tilting modules $\approx$ bases in linear algebra
for abelian categ

Prop | number of
\# tilting modules in mod kQ
for Q an orient of o–o–o–o–o (n vert)
is the Catalan number C_{n-1} $\forall n$

not going to prove that (immediatly)

Prop for $Q = o \to o \to o \to \cdots \to o$ ($\vec{A}_n$)
there is a simple bijection {tilting modules}
$\simeq$ {planar binary trees}

A.R quiver look like that

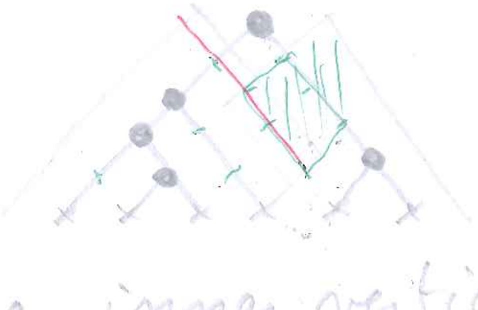


Ext $\neq 0$ zones

so tilting modules $\leftrightarrow$ maximal configurations
of compatible points

---

on the other hand, draw planar binary trees
upside down with leaves aligned horizontally



then inner vertices satisfy $\nearrow$ Ext $= 0$
$\searrow$ maximality
(by induction) hence define tilt modules

partial order on the set of tilting modules

mutation of tilting modules

in any tilting module $T = T_1 \oplus \cdots \oplus T_n$ (like in bases)

n is always the same integer $= |Q_0|$

number of vertices of the quiver

for example all projective or all injectives

Def almost-complete tilting module = partial tilting module (with $T_i \neq T_j$)

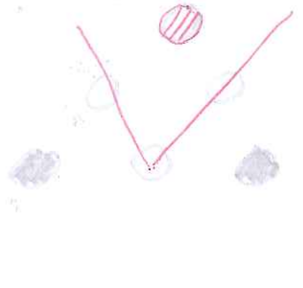
$T = T_1 \oplus \cdots \oplus T_{n-1}$ (missing one summand)

morally = missing one indecomp. to be tilting

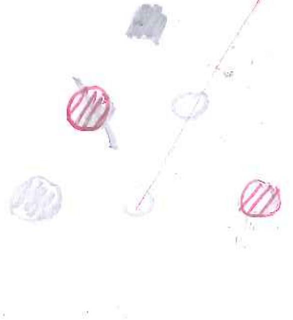
Prop there are at most 2 ways to complete an almost complete tilt. mod into a tilt module

There can be just one complement, but not 0.

Example



almost complete with one complem.



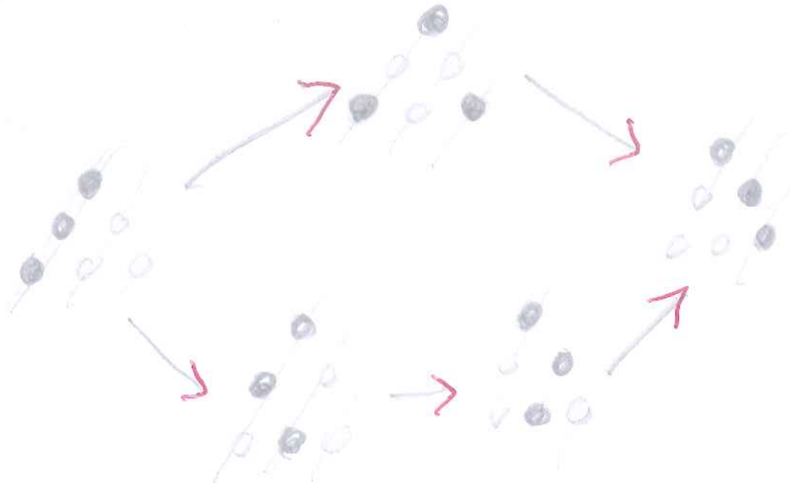
with 2 complements

claim : graph { - tilting modules
- mutation

can be oriented and gives an acyclic oriented graph hence a partial order

(Riedtmann – Schofield)

(known to Gabriel)?



orientation = being more to the right

exact meaning * $\forall\, T_{\text{ind}}\ \exists\, n(T)$ s.t $\tau^{n(T)} T$ proj

* if T' and T'' are two complements then $n(T') \neq n(T'')$. . .

full subcateg. closed by
- Ext
- isom
- factor

Another way to define the same order

{tilting modules} $\xrightarrow{\text{inject}}$ torsion classes

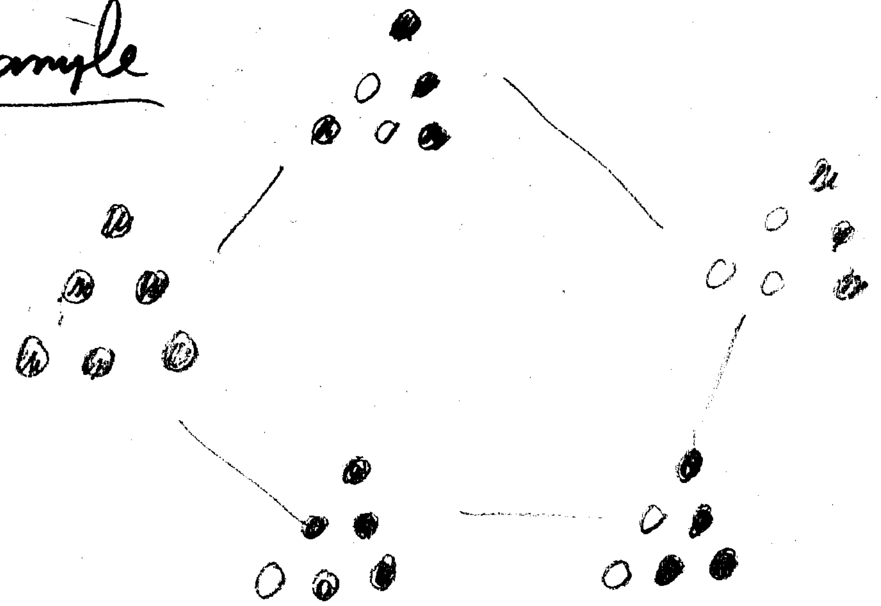
$T \longrightarrow$ Fac(T) = subcategory of factor modules of direct sums of T

ordered by inclusion

(This is even a bijection in our context)

<u>Example</u>

torsion classes as sets of indecomp.



<u>Prop</u> : for the quiver $\overrightarrow{A_n} = \circ \to \circ \to \quad \to \circ$

the poset of tilting modules $\simeq$ the Tamari lattice of planar binary trees

Id. Est: mutation of tilting mod $\longleftrightarrow$ rotation of planar bin. trees

---

<u>So</u> summary

Quiver $\overrightarrow{A_n}$ $\circ\to\circ\to\circ\to\circ$ $\rightsquigarrow$ c_n tilting mod $\rightsquigarrow$ Tamari order on bin. trees

other orientations $\rightsquigarrow$ c_n ——— $\rightsquigarrow$ ~~Cambrian lattices of type A~~ other posets

Later : all share something