Triangulating the Permutahedron.

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The Question

For an Artin group $A(W)$ of finite type W we have a pair of $K(A(W), 1)$'s.

One is given by making identifications on the boundary of the type W permutahedron. (Salvetti $K(A(W), 1)$)

The other is given by making identifications on the boundary of order complex of the type W NCP lattice. (NCP $K(A(W), 1)$)

Since they have the same homotopy type there should be a homotopy equivalence between them. Is there a closed formula?

The Salvetti $K(A(W), 1)$

Consider the type W permutahedron P.

P is obtained by taking the convex hull in $\mathbb{R}^n$ of the translates under W of a single point in the interior of some chamber.

The edges at each vertex are labelled by simple reflections.

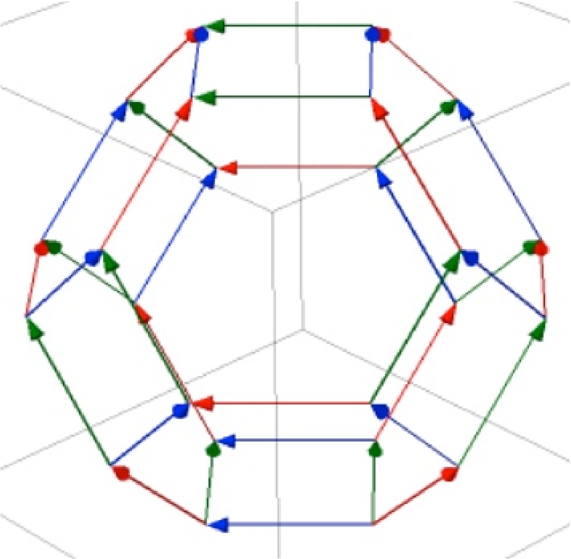
Direct the edges from some chosen identity vertex e towards the longest element vertex w_0.

Define an identification space $P/\sim$, where cells on the boundary with the same labels are identified.

The resulting cell complex satisfies $\pi_1(P/\sim) \cong A(W)$ and $\pi_k(P/\sim) \cong 0$ for $k > 1$.

Call $P/\sim$ the Salvetti $K(A(W), 1)$.

The Salvetti $K(A(W), 1)$



The NCP $K(A(W), 1)$

For $w \in W$, define $|w|$ to be the total reflection length of w and define a partial order $\leq$ on W by $w_1 \leq w_2$ if and only if $|w_2| = |w_1| + |w_1^{-1} w_2|$.

Fix a Coxeter element γ and define NCP(W), the lattice of type W non-crossing partitions to be the set of elements in $[e, \gamma]$ ordered by $\leq$. Define $Q = |NCP(W)|$ to be the order complex of $NCP(W)$.

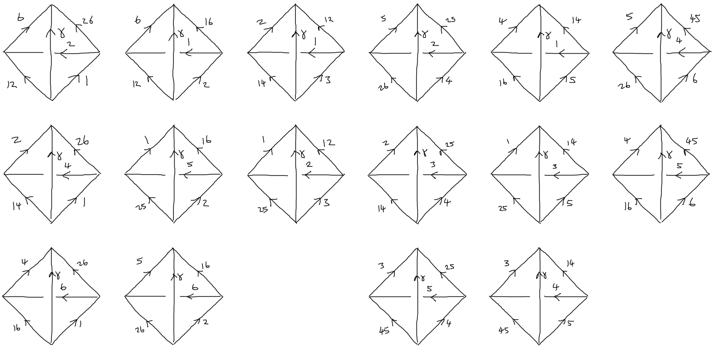
As with the permutahedron, identify faces of Q with the same labels.

The resulting cell complex also satisfies $\pi_1(Q/\sim) \cong A(W)$ and $\pi_k(Q/\sim) \cong 0$ for $k > 1$.

Call $Q/\sim$ the NCP $K(A(W), 1)$.

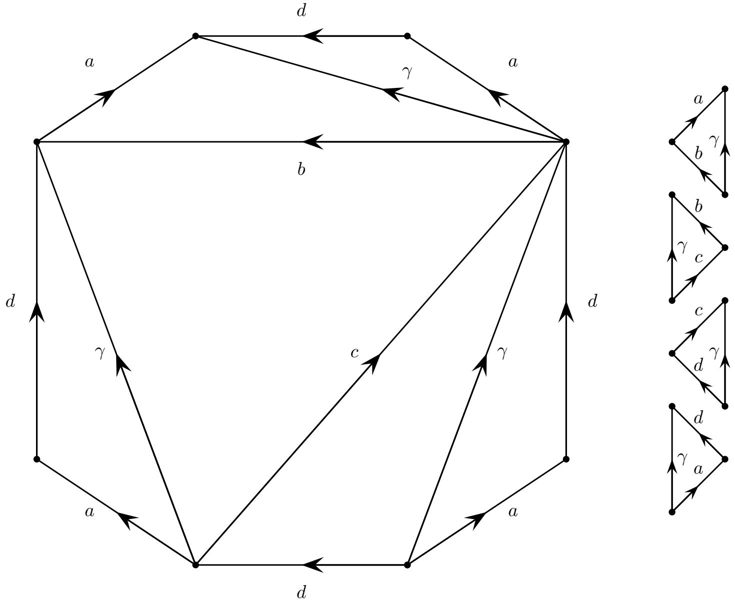
NCP example, $W = \Sigma_4$

Here reflections are transpositions, $|w|$ is transposition length, $\gamma = (1,2,3,4)$ 4-cycle, $w \le \gamma$ if and only if blocks of w do not cross and cyclic order in blocks is that induced by γ. Q has 16 3-cells corresponding to the 16 ways of factoring a 4-cycle as a product of 3 transpositions.



$1 : (1,2), \quad 2 : (3,4), \quad 3 : (1,3), \quad 4 : (2,3), \quad 5 : (1,4), \quad 6 : (2,4)$

Dihedral Sal to NCP map



Bruhat Intervals

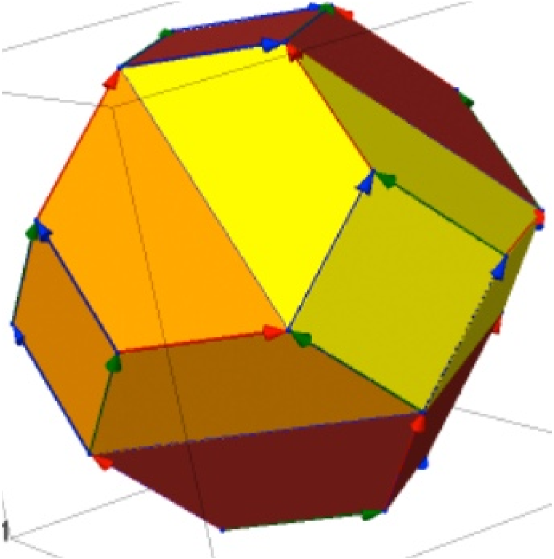
Let v_0 be the chosen point in the interior of the fundamental (identity) chamber.

Definition: Declare a facet on the ordered subset $(w_0(v_0), w_1(v_0), \dots, w_n(v_0))$ of $\mathbb{R}^n$ if and only if
(i) $l_S(w_i) = l_S(w_0) + i$ and
(ii) $e \lessdot w_0^{-1}w_1 \lessdot \cdots \lessdot w_0^{-1}w_n$ is a maximal chain in the NCP lattice.
In particular, each $w_i^{-1}w_{i+1}$ is a reflection and $w_0^{-1}w_n = \gamma$.

Note: Thus $w_0 < w_1 < \cdots < w_n$ is a maximal chain in the Bruhat interval $[w_0, w_0\gamma]$.

Note: In general a collection of maximally independent subsets of a point set in $\mathbb{R}^n$ (a) may not exhaust the convex hull and (b) may overlap in their interiors.

Permutahedron



Crash course on W-associahedron

W frrg of rank n acting on $\mathbb{R}^n$

C fundamental domain for W action

$\{\alpha_1, \ldots, \alpha_n\}$ inward unit normals to walls of C

$R_1, \ldots, R_n$ corresponding reflections

$\gamma = R_1 R_2 \ldots R_n$, a Coxeter element with order h

Define roots $\rho_i = R_1 \ldots R_{i-1}\alpha_i$, where $R_{n+j} := R_j$, $\alpha_{n+j} := \alpha_j$

Let $\{\beta_1, \ldots, \beta_n\}$ be dual basis to $\{\alpha_1, \ldots, \alpha_n\}$

Define $\mu_i = \mu(\rho_i) = R_1 \ldots R_{i-1}\beta_i$, where $R_{n+j} := R_j$, $\beta_{n+j} := \beta_j$

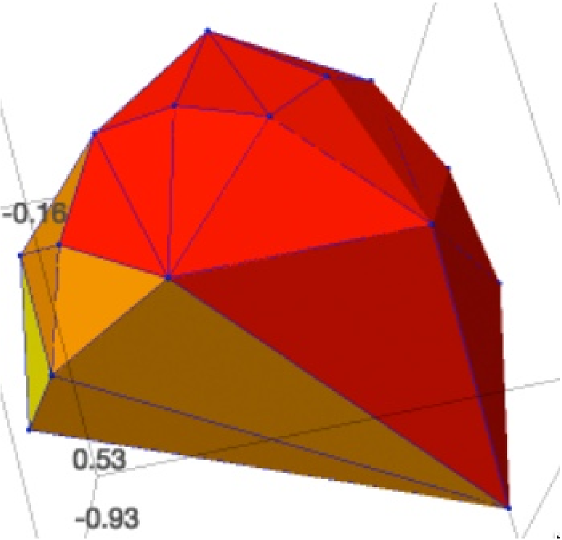
Define $\mu A(X(\gamma))$ to be simplicial complex with vertex set

$$\mu_1, \mu_2, \ldots, \mu_{nh/2}, \mu_{nh/2+1}, \ldots, \mu_{nh/2+n}$$

and a simplex on $\{\mu(\tau_1), \ldots, \mu(\tau_k)\}$ provided

$\rho_1 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq \rho_{nh/2+n}$ and $|R(\tau_1) \ldots R(\tau_k)\gamma| = n-k$.

W-associahedron for icosahedral W



Proposition: Fix an element $w_0 \in W$ and positive roots $\theta_1, \ldots, \theta_n$ with $\gamma = R(\theta_1)R(\theta_2)\cdots R(\theta_n)$. Define the elements $w_j = w_0 R(\theta_1)R(\theta_2)\cdots R(\theta_j)$ for $0 \le j \le n$. Then $(w_0, w_1, \ldots, w_n)$ is a facet $\Leftrightarrow$ $w_n^{-1}(v_0) \in \mathrm{cone}(\{-\mu(\theta_1), -\mu(\theta_2), \ldots, -\mu(\theta_n)\})$.

Proof: $(\Rightarrow)$ The maximal chain condition implies $w_j^{-1}(v_0)\cdot\theta_j < 0$. Define the root τ_j by $\tau_j = R(\theta_n)\cdots R(\theta_{j+1})\theta_j$ for $1 \le j \le n$. $w_n^{-1}(v_0) = a_1\mu(\theta_1) + \cdots + a_n\mu(\theta_n)$, $\exists! a_1, \ldots, a_n$.

$$\begin{aligned}
0 &> w_j^{-1}(v_0)\cdot\theta_j \\
&= w_n^{-1}(v_0)\cdot\tau_j \\
&= [a_1\mu(\theta_1) + \cdots + a_n\mu(\theta_n)]\cdot\tau_j \\
&= a_j\mu(\theta_j)\cdot\tau_j \qquad (\text{by } (*)) \\
&= a_j\mu(\theta_j)\cdot R(\theta_n)\cdots R(\theta_{j+1})\theta_j \\
&= a_j[R(\theta_{j+1})\cdots R(\theta_n)\mu(\theta_j)]\cdot\theta_j \\
&= a_j\mu(\theta_j)\cdot\theta_j \qquad (\text{by } (**)) \\
&= a_j
\end{aligned}$$