

Fourier Analysis

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Abstract:

The aim of this experiment was to investigate the Fourier transforms of periodic waveforms, and using harmonic analysis of Fourier transforms to gain information about the frequencies present. Sine waves, square waves and rectangular waves were created using an oscillator and a speaker in air and their Fourier transforms obtained and compared with those predicted by the theory. These methods were used to determine the range of human hearing, which was found to be $108\text{Hz} \pm 1\text{Hz}$ to $18,450\text{ Hz} \pm 50\text{Hz}$, and also to calculate the beat frequency of two tuning forks.

Aims:

Our aims in this experiment were;

- To gain an intuitive understanding of Fourier transforms
- To analyse various waveforms and explain their Fourier transforms
- To find a beat frequency of two tuning forks
- To investigate the thresholds of human hearing
- To compare human voices, along with different vowel sounds

Introduction and Theory:

Fourier analysis of periodic waves is a method of decomposing periodic functions (ie. $f(x + l) = f(x)$) into an infinite sum of the trigonometric functions sin and cos, known as the Fourier series, like so:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$a_0 = \frac{1}{p} \int_{-p/2}^{p/2} f(x) dx$$

$$a_n = \frac{2}{p} \int_{-p/2}^{p/2} f(x) \cos\left(\frac{2\pi nx}{p}\right) dx$$

$$b_n = \frac{2}{p} \int_{-p/2}^{p/2} f(x) \sin\left(\frac{2\pi nx}{p}\right) dx$$

These are known as the Euler formulae for the Fourier coefficients, and the coefficients a_n and b_n give an idea of how much of any particular frequency is present in a function's Fourier series.

The first formula given above is known as the simple Fourier series, which can then be extended to the complex Fourier series with complex coefficients. From this, we get the notion of the Fourier transform, which is a mathematical operation which decomposes a function into its constituent frequencies, which are analogous to the Fourier coefficients as before. The complex Fourier transform is given by the formula

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi ift} dt$$

Where $\hat{x}(f)$ is the Fourier transform of our function $x(t)$, f is frequency, and t is time.

Fourier analysis can therefore be used to determine the amount of any given frequency in a function, and given a signal, it can be used to create a frequency spectrum of this signal.

In this and other experiments, a discrete Fourier transform is used, which obtains the Fourier transform of a function by sampling discrete data, ie. approximating a continuous function by collecting data at certain intervals. In this experiment the software Cassy lab is used, and the Fast Fourier Transform algorithm is used to make the process more rapid.

Experimental Method:

Part 1:

- The correct parameters and settings were applied using the Cassy lab program.
- The graphs of the function and the Fourier transform of a sine wave were obtained and saved.
- The graphs of the function and the Fourier transform of the function given in the objective sheet were obtained and saved. The function was identified.

Part 2:

- The oscillator was connected to a speaker, and the microphone to the input device.
- A square wave was generated in the air and its graphs recorded.
- The upper and lower thresholds of hearing were measured.
- A beat frequency was measured using tuning forks.
- Different voices were compared and their fundamental frequencies found. Also the spectrum of different vowels was examined.

Part 3:

- A sine wave was generated and by increasing the frequency the Fourier transform was observed moving across the screen.

Part 4:

- A single isolated rectangular pulse was created and its spectrum recorded.

- The frequencies at which the first three minima appeared were recorded, along with the amplitudes at $f=0$ and the next two maxima.

Part 5:

- A square pulse was created.
- The positions of the first three minima and the amplitudes of the first three maxima were recorded.
- A sine pulse was then created and the spectra recorded.

Results and Analysis:

Part 1:

By Euler's Theorem, a sine wave can be represented as

$$\sin(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} - e^{-i2\pi f_0 t}}{2i}$$

We then have the Fourier transform of this sine wave:

$$\begin{aligned}\hat{x}(f) &= \int_{-\infty}^{\infty} \frac{e^{i2\pi f_0 t} - e^{-i2\pi f_0 t}}{2i} \cdot e^{-2\pi i f t} dt \\ &= \frac{1}{2i} (\delta(f - f_0) - \delta(f + f_0))\end{aligned}$$

Where $\delta(x)$ is the Dirac Delta function. Since a sine wave consists of only one frequency we have $f = f_0$ and the Fourier transform has a peak at f_0 only, which we can see from the graph below.

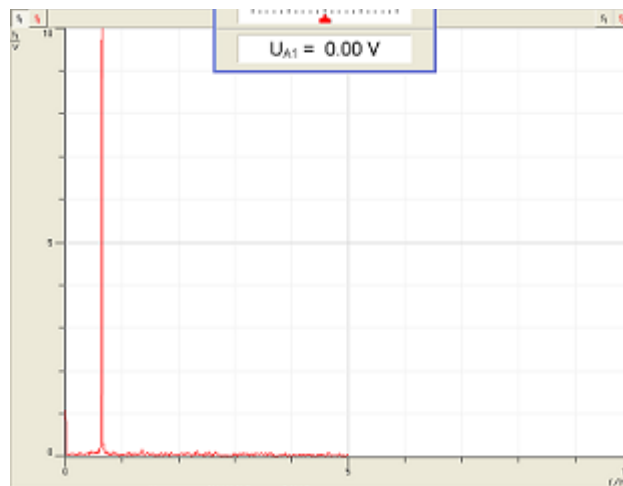


Fig. 1(a) Fourier transform of a sine wave.

We are then given the function

$$f(x) = \begin{cases} -L, & -L < x < 0 \\ L, & 0 < x < L \end{cases}$$

Where $f(x) = f(x + 2L)$. This defines a square wave. We need to show that only odd harmonics are present, and to do this we use the fact that our function is odd, ie. $f(-x) = -f(x)$. Let us examine the b_n coefficients of our function.

$$\begin{aligned}
 b_n &= \frac{2}{p} \int_{-p/2}^{p/2} f(x) \sin\left(\frac{2\pi nx}{p}\right) dx \\
 &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{\pi nx}{L}\right) dx
 \end{aligned}$$

We can split this integral into two integrals, as follows

$$\begin{aligned}
 &= \frac{1}{L} \left[\int_0^L f(x) \sin\left(\frac{\pi nx}{L}\right) dx + \int_{-L}^0 f(x) \sin\left(\frac{\pi nx}{L}\right) dx \right] \\
 &= \frac{1}{L} \left[\int_0^L L \sin\left(\frac{\pi nx}{L}\right) dx + \int_{-L}^0 -L \sin\left(\frac{\pi nx}{L}\right) dx \right] \\
 &= \frac{1}{L} \left[\int_0^L L \sin\left(\frac{\pi nx}{L}\right) dx + \int_0^L L \sin\left(\frac{\pi nx}{L}\right) dx \right] \\
 &= 2 \left[\int_0^L \sin\left(\frac{\pi nx}{L}\right) dx \right] \\
 &= \frac{2L}{\pi n} [1 - \cos(\pi n)]
 \end{aligned}$$

But we know, $1 - \cos(\pi n) = \begin{cases} 2 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

So we have,

$$b_1 = \frac{2L}{\pi}, b_2 = 0, b_3 = \frac{4L}{3\pi}, \dots$$

This shows that only odd harmonics are non-zero in the Fourier transform, which is verified by the graph below.

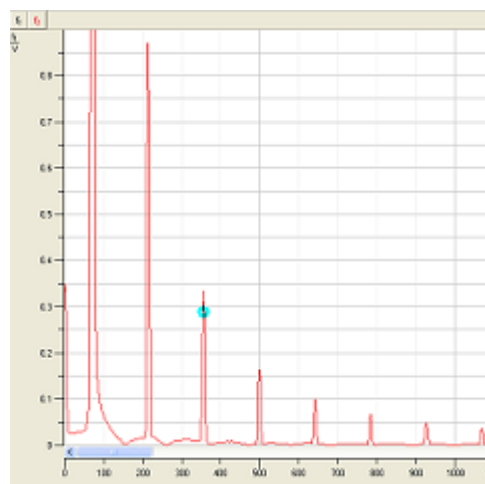


Fig.1(b) Fourier transform of a square wave

Part 2:

A square wave was created in air and it was observed that the spectrum was less distinct than that of the one not in air, due to the variations of pressure waves in air caused by various effects, like the viscosity of the medium, the temperature and the movement of the air itself.

The lower threshold of human hearing was found to be $108\text{Hz} \pm 1\text{Hz}$, and the upper threshold of hearing was found to be $18,450\text{ Hz} \pm 50\text{Hz}$.

The beat frequency of the tuning forks was found to be approximately 20Hz , which agrees with the theoretical value of $f_a - f_b$, where $f_a = 384\text{Hz}$, and $f_b = 361\text{Hz}$. The spectrum of the beat frequency was quite similar to the spectrum of the coupled pendulum as shown below.

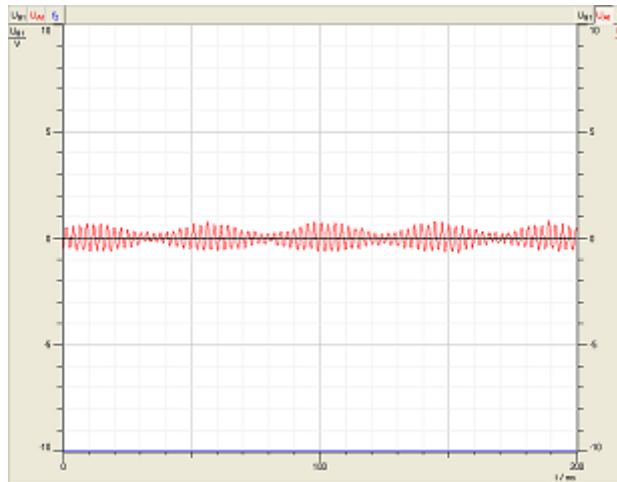


Fig. 2(a) Spectrum of the beat frequency

The spectra for human voices were compared, and the fundamental frequency of a male voice was found to be $f_m = 485 \pm 1\text{Hz}$, and the fundamental frequency of a female voice was found to be $f_f = 494 \pm 1\text{Hz}$. This agrees with the intuitive idea that the frequency of a male voice is lower than that of a female voice. Two different vowel sounds were also compared, that of the letter “a” and the letter “o”, and it was observed that their Fourier transforms were quite different, implying that there were different amounts of various frequencies present in each.

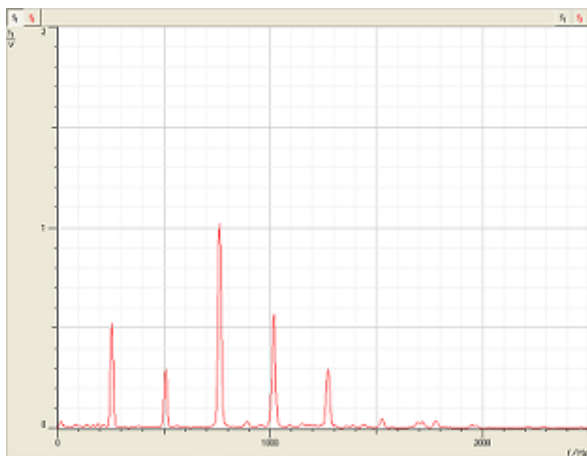


Fig. 2(b) Fourier transform of vowel a

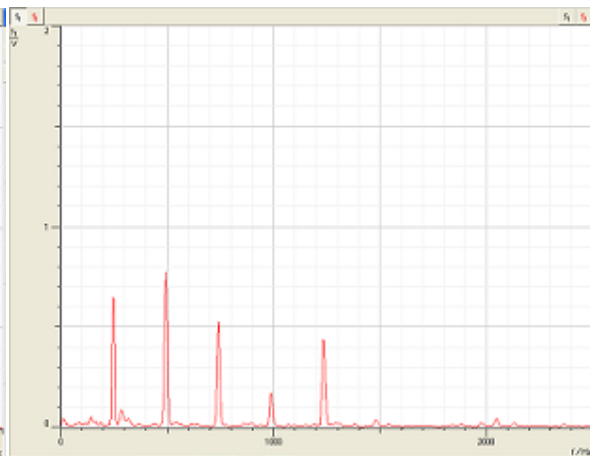


Fig. 2(c) Fourier transform of vowel o

Part 3:

As the frequency of the input sine wave was increased, the Fourier transform moved to the right across the screen. However, once the frequency had reached a threshold such that $f_{input} > \frac{f_{sampling}}{2}$ the wave became aliased, ie the program attempted to reconstruct the signal under the alias of another signal, assuming it has a frequency less than $\frac{f_{sampling}}{2}$. Once this occurred the Fourier transform appeared on the left and began to move to the right across the screen once again.

Part 4:

A single rectangular pulse is given by the formula

$$f(x) = \begin{cases} V, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

Since $f(x)$ is an even function, the sine coefficients are zero, so we use the Fourier cosine transformation given by

$$\begin{aligned} \hat{x}(f) &= \sqrt{\frac{2}{\pi}} \int_0^{L/2} f(x) \cos(2\pi f x) dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{L/2} V \cos(2\pi f x) dx \\ &= V \sqrt{\frac{2}{\pi}} \left[\frac{\sin(2\pi f x)}{2\pi f} \right] \\ &= V \sqrt{\frac{2}{\pi}} \left[\frac{\sin(\pi f L)}{2\pi f} \right] \\ &= V \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\pi f L)}{\pi f} \right] \end{aligned}$$

We know that $f(x)$ is zero when $x \neq L/2$, so these minima must occur when $\sin(\pi f L) = 0$, so $\pi f L = n\pi$, ie $f = n/L$. This means that the minima must occur at frequencies which are integer multiples of the frequency of the first minimum. We measured the first three minima to occur at the frequencies $f_1 = 1.37 \pm 0.25\text{Hz}$, $f_2 = 2.83 \pm 0.25\text{Hz}$, and $f_3 = 4.25 \pm 0.25\text{Hz}$ which agree very well with the theory.

Using the approximation $\frac{\sin\theta}{\theta} \cong 1$ as θ approaches zero, we can make the approximation

$$A_1 \cong \frac{VL}{\sqrt{2\pi}}$$

Since the maxima occur at $\sin(\pi f L) = 1$, we get $\pi f L = \frac{n\pi}{2}$ and $f = \frac{\pi}{2L}$ where n is odd.

So our original formula becomes

$$A_n = V \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\pi f L)}{\pi f} \right] = \frac{V}{\sqrt{2\pi}} \cdot \frac{1}{\pi \left(\frac{\pi}{2L} \right)} = \frac{2}{n\pi} A_1$$

The values we measured for the maxima occur at $V_1 = 0.55 \pm 0.01V$, $V_2 = 0.11 \pm 0.01V$, $V_3 = 0.06 \pm 0.01V$, which again agree perfectly with the theoretical values. When we varied the length of the pulse, we discovered that the length of the pulse was inversely proportional to the spacing between successive zeroes.

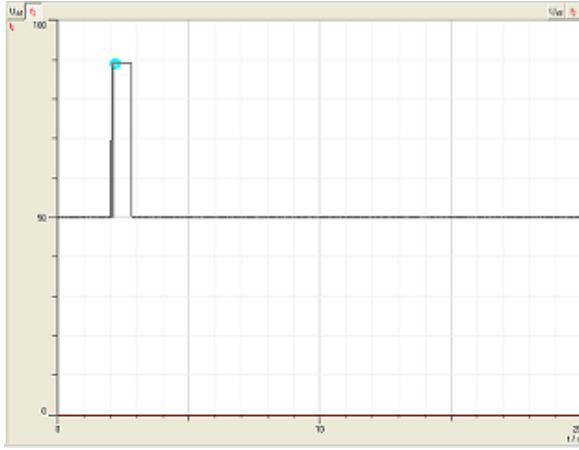


Fig. 4(a) A rectangular pulse

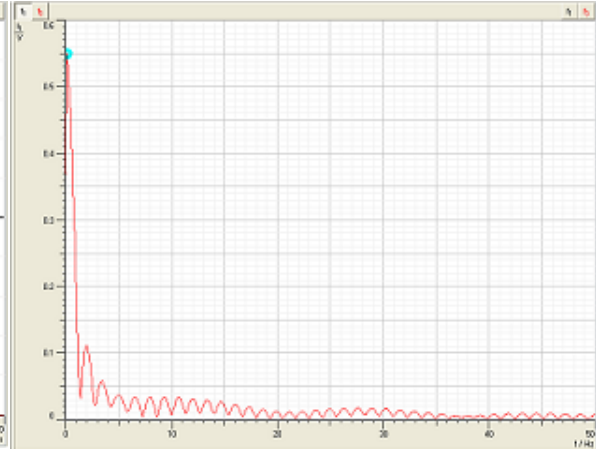


Fig. 4(b) Fourier transform of a rectangular pulse

Part 5:

A single square wave pulse given by the formula

$$f(x) = \begin{cases} L, & |x| < L/2 \\ 0, & |x| > L/2 \end{cases}$$

This is identical to the rectangular pulse except for $V=L$. The theory behind locations of minima and maxima remains the same, so our minima frequencies of $f_1 = 40.5 \pm 1\text{Hz}$, $f_2 = 80.1 \pm 1\text{Hz}$, and $f_3 = 122.9 \pm 1\text{Hz}$, and our maxima at $V_1 = 0.3 \pm 0.01V$, $V_2 = 0.1 \pm 0.01V$, $V_3 = 0.06 \pm 0.01V$ also agree very well with our theoretical values.

A sine wave pulse is given by

$$f(x) = \begin{cases} V \cos(\omega x), & -L, x, L \\ 0, & |x| > L \end{cases}$$

All of the sine coefficients are zero, meaning we need only consider the cosine coefficients, leaving

$$a_n = \int_{-L}^L V \cos(\omega x) \cos(nx) dx$$

Using trigonometry we get

$$a_n = \frac{V}{2} \int_{-L}^L [\cos((\omega + n)x) + \cos((\omega - n)x) dx]$$

$$\begin{aligned}
&= \frac{V}{2} \left[\int_{-L}^L \cos((\omega + n)x) dx + \int_{-L}^L \cos((\omega - n)x) dx \right] \\
&= VL \left[\frac{\sin((\omega + n)L)}{L(\omega + n)} + \frac{\sin((\omega - n)L)}{L(\omega - n)} \right] \\
&\therefore a_n = VL[\text{sinc}(\omega + n)L + \text{sinc}(\omega - n)L]
\end{aligned}$$

We expect the Fourier transform to be simply a peak at the frequency of the pulse, similar to the continuous sine wave we had before, as we can see below.

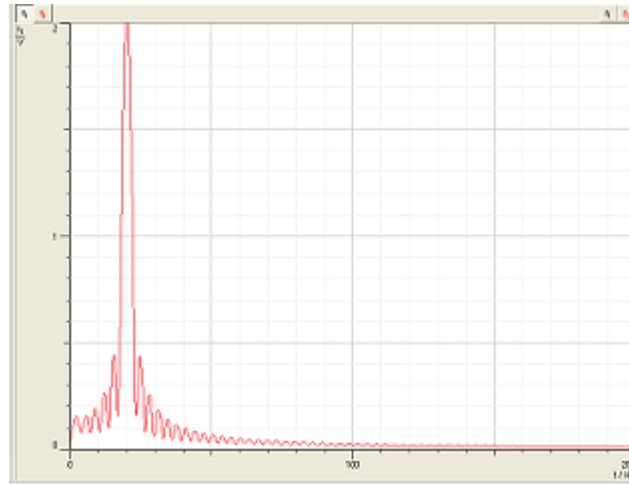


Fig 5. Fourier transform of a sine wave pulse

Discussions and Conclusions:

- The Fourier transforms of many waves were investigated, recorded and explained, including those of sine waves and square waves.
- The Fourier transforms of wave pulses were investigated, recorded and explained, including rectangular pulses, square pulses, and sine pulses.
- The lower threshold of human hearing was found to be $108\text{Hz} \pm 1\text{Hz}$, and the upper threshold of hearing was found to be $18,450\text{ Hz} \pm 50\text{Hz}$.
- The fundamental frequency of a male voice was found to be $f_m = 485 \pm 1\text{Hz}$, and the fundamental frequency of a female voice was found to be $f_f = 494 \pm 1\text{Hz}$.
- The beat frequency of two tuning forks was found to be approximately 20Hz , which agrees with the theoretical value of $f_a - f_b$, where $f_a = 384\text{Hz}$, and $f_b = 361\text{Hz}$.
- Overall a working knowledge of Fourier transforms was gained, and the experiment was regarded as highly successful as the experimental data verified the theory throughout.
- Unfortunately once the experiment was completed, it was discovered that some of the graphs of functions and Fourier transforms had saved incorrectly, by which time it was too late to attempt to re-do them.