Integrable Form Factors and AdS/CFT

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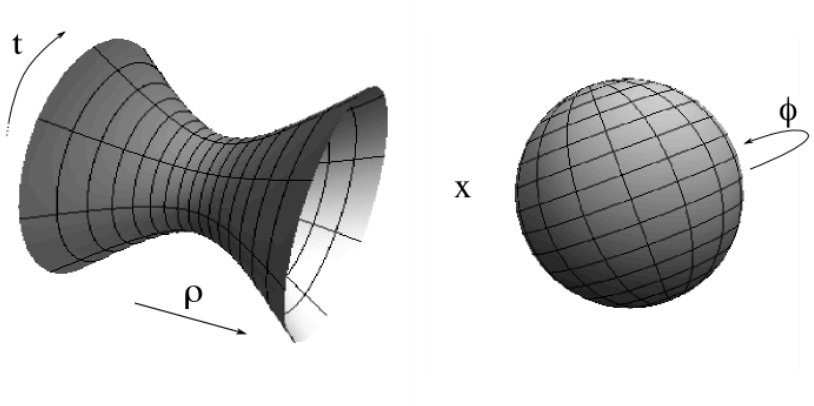
Based on work with T. Klose, L. Gerotto.

AdS$_5$ *x* S^5 string theory

Starting point is the GS string on AdS$_5$ *x* S^5:



In (uniform) light-cone gauge

$$\tilde{\tau}(\tau,\sigma) = X^+ \equiv (1-a)t + a\phi \ , \quad p_+ = (1-a)p_\phi - ap_t = \text{constant}$$

theory is a massive, integrable, non-Lorentz invariant 1+1-dimensional theory defined on a cylinder:

$$\frac{L}{2\pi} = (1-a)\mathcal{J} + a\mathcal{E}$$

$$J = \sqrt{\lambda}\mathcal{J}$$

$$E = \sqrt{\lambda}\mathcal{E}$$

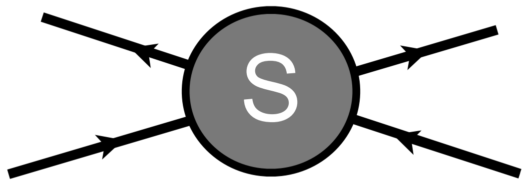
World-sheet Scattering

In the de-compactified limit one can define asymptotic incoming states and so "in"-scattering states

$$|p_1, p_2, \ldots, p_n\rangle^{(\mathrm{in})}_{i_1,i_2,\ldots,i_n} \equiv |p_1, p_2, \ldots, p_n\rangle_{i_1,i_2,\ldots,i_n}$$

with $p_1 > p_2 > \cdots > p_n$ and reversed for "out"-scattering states.

The "in"- and "out"-scattering states are then related by the S-matrix



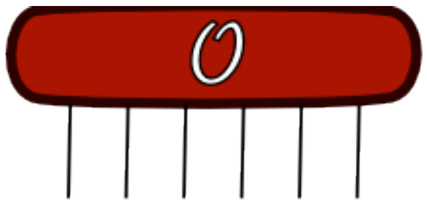
$$|p_1, p_2\rangle^{(\mathrm{in})} = S_{12}(p_1, p_2) \cdot |p_1, p_2\rangle^{(\mathrm{out})}$$

Combined with a knowledge of the symmetries and bound states this allows for a complete determination of the S-matrix which is basic building block for determining string energies.

World-sheet Form Factors

We are interested in calculating matrix elements of world-sheet operators between asymptotic states.

We define auxiliary functions as matrix elements for "in"-scattering state (i.e. with $p_1 > p_2 > \ldots > p_n$)



$$f_{\underline{i}}^{\mathcal{O}}(p_{\underline{i}}) = \langle\Omega|\mathcal{O}|p_1, \ldots, p_n\rangle^{(\text{in})}_{i_1,\ldots,i_n}$$

and for other values of the external momentum by analytic continuation

Can find other matrix elements by using crossing.

World-sheet Correlation Functions

- In particular they can be used to construct world-sheet correlation functions which by the AdS/CFT duality are related to gauge theory higher-point functions.

- We are most interested in the case when the world-sheet operators are related to string vertex operators which create some state dual to a “short” gauge theory operator while the asymptotic states will correspond to near-BPS operators.

- In this case world-sheet form factors can be directly related to gauge theory structure constants (for certain operators, in certain limits ...).

- We can see this concretely by considering the spin-chain description of tree-level structure constants.

Spin-chain form factors

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle = \frac{c_{abc}}{|x_{12}|^{\alpha_c}|x_{23}|^{\alpha_a}|x_{31}|^{\alpha_b}}$$

At weak coupling we can map the calculation of tree-level structure constants to the calculation of spin-chain form factors for the case where two operators have the same length.

$$c^0_{abc} \propto \langle \Psi_{L_c}(\mu)|\sigma^z_1|\Psi_{L_c}(\lambda)\rangle$$

Spin-chain form factors

Ex. [Roiban & Volovich] Operators made from single traces of two types of complex scalars

$$\mathcal{O} \sim \sum \cos \frac{\pi n(2s+1)}{L_c} \mathrm{Tr} \left[Y (Z)^s Y (Z)^{L_c - 2 - s} \right]$$

at one-loop the dilatation operator is described by XXX-spin chain

$$H = \frac{\lambda}{16\pi^2} \sum_{x=1}^{L_c} (1 - \vec{\sigma}_x \cdot \vec{\sigma}_{x+1})$$

and eigen-operators can be described by Bethe states.

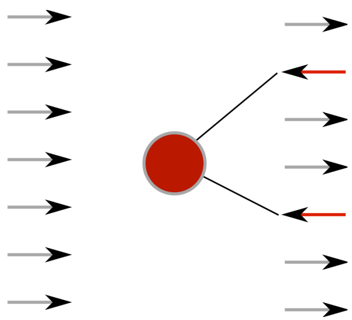
Spin-chain form factors

For the gauge theory/spin-chain operator

$$\mathcal{O}_b = \mathrm{Tr}(ZZ\bar{Y}\bar{Y}) \leftrightarrow \mathcal{O}_b = \sum_{j=1}^{L_c} S_{+,j}S_{+,J+1}$$

Consider

$$O_a \sim \mathrm{Tr}(\bar{Z}^{L_c})$$



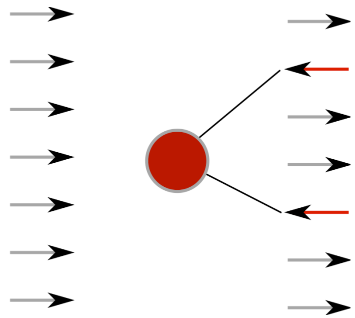
$$O_c \sim \sum_j \cos\frac{\pi n(2j+1)}{L_c-1}\mathrm{Tr}(YZ^iYZ^{L_c-2-j})$$

$$c^0_{abc} \sim f_{\text{spin}}(p_1,p_2) = {}_{L_c}\langle 0|S_{+,j}S_{+,j+1}|p_1,p_2\rangle_{L_c}$$

Let momenta satisfy B.E. but relax level matching
a.k.a trace cyclicity

Weak-strong matching

Example: in the limit $L_c \to \infty$ so that $p_i = \dfrac{2\pi n_i}{L_c} + \mathcal{O}(1/L_c^2)$ then for generic momenta and to next-to-leading order



$$f_{\rm spin}(p_1, p_2) = \frac{2}{L_c} + \frac{6i\pi}{L_c^2}(n_1 + n_2) + \frac{2}{L_c^2}\frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + \mathcal{O}(L_c^{-3})$$

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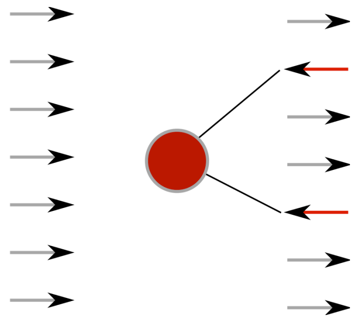

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Wave-function Phase Wave-function Norm

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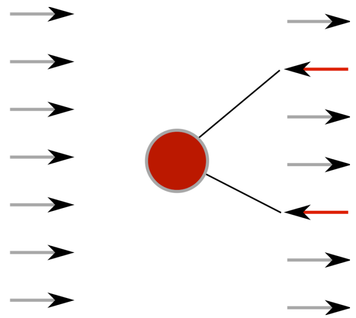

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Wave-function Phase Wave-function Norm

Can this be matched with string theory?

Weak-strong matching

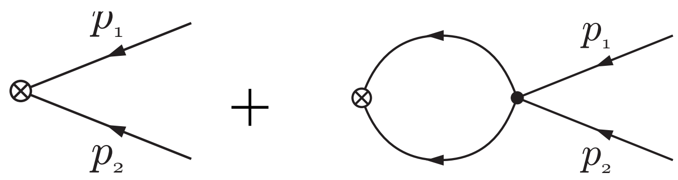
To see matching explicitly.

Procedure:

- Match spin-chain operator to string theory operator.
- Calculate form factor in the de-compactified limit.
- Re-compactify form factor by
  - imposing Bethe-Yang equations.
  - including correct normalisation (c.f. [Pozsgay and Takacs] finite-volume form factors).

Two-point form factor

The string theory object is the two-point function (for $\mathcal{O} = Y^2$) which is an essential building block for all other higher point form factors.



Tree-level:

$$f^{(0)}(p_1, p_2) = \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}}$$

One-loop:

$$f^{(1)}_{\text{ren}}(\theta_1, \theta_2) = \frac{1}{\sqrt{\lambda}} \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\theta - i\pi) \Big[\coth \frac{\theta}{2} \sinh^2 \frac{\tilde{\theta}}{2} + \frac{1-2a}{2} \sinh \theta \Big]$$

Note:

- Y^2 operator mixes with a two-derivative Lorentz scalar operator

Form factor normalisation

To match with the spin-chain we fix the momenta in terms of mode numbers using the string B.E. for example

$$p_1 = \frac{2\pi n_1}{L_s} - \frac{4\pi^2}{\sqrt{\lambda} L_s^2} \frac{n_1^2 + n_2^2 - a(n_1 - n_2)^2}{n_1 - n_2} + \mathcal{O}(\lambda^{-1})$$

We also need to ensure the states are normalised equivalently. The perturbative calculation of the form factor is done for on-shell states with the inner product

$$\langle p_3, p_4 | p_1, p_2 \rangle = (2\pi)^2 \left[\delta(p_1 - p_3)\delta(p_2 - p_4) + \text{crossed channel} \right]$$

whereas to match the spin-chain they should be in terms of mode-number delta-functions. The Jacobian for the change of variables is

$$\mathcal{N}_{\mathrm{s}} = (2\pi)^2 \left| \frac{\partial(p_1, p_2)}{\partial(n_1, n_2)} \right|^{-1}$$

which is essentially the Gaudin norm for Bethe state.

Combining these factors, using the relation between the string length and spin-chain length in $a = 1$ gauge, and allowing for the splitting of the spin-chain operators we find for the world-sheet form factor

$$\hat{f}_{\text{ws}}(n_1, n_2) = \frac{2}{L_c} + \frac{2i\pi}{L_c^2}\left(\frac{2n_1 n_2}{n_1 - n_2} - (n_1 + n_2)\right) + \frac{2}{L_c^2}\frac{n_1^2 + n_2^2}{(n_1 - n_2)^2}$$

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Matches Spin-Chain

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Corresponds to a wave-function

$$\sim e^{-2i(p_1+p_2)}\chi_{\rm spin}(p_1, p_2)$$

Do we expect this agreement to persist? No.

This is in the double scaling limit:

$$f_{\rm string}(\lambda, J)\Big|_{\substack{\lambda\to\infty \\ J\to\infty \\ \tilde{\lambda}\to 0}} \stackrel{?}{=} f_{\rm spin}(\lambda, J)\Big|_{\substack{\lambda\to 0 \\ J\to\infty \\ \tilde{\lambda}\to 0}}$$

So agreement is due to some unknown non-renormalization theorem (presumably the same as that which gives matching of anomalous dimensions).

To get proper matching one should find all order answers.

Integrable Form Factors

This may be possible as due to unitarity, analyticity and locality there are a system of functional equations which when combined with integrability might allow form factors for this theory to be exactly calculable

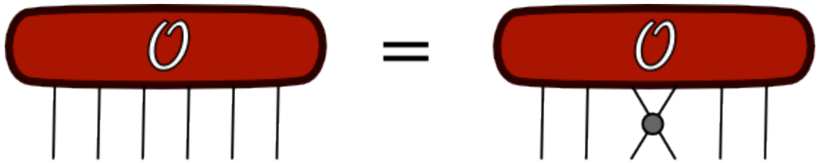
Form factor Axioms

Define

$$f^{\mathcal{O}}(z_1,\ldots,z_n) = \langle \Omega | \mathcal{O} | p(z_1),\ldots p(z_n)\rangle\ , \quad p(z_1) > \cdots > p(z_n)$$

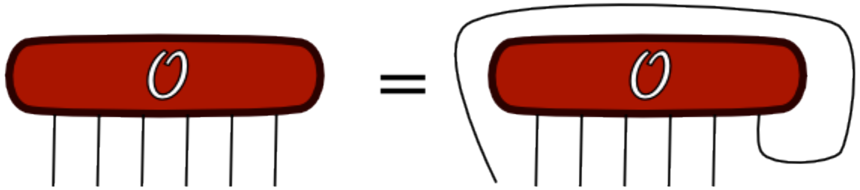
<u>Axioms</u>

i) Permutation:



$$f^{\mathcal{O}}(\ldots,z_j,z_i,\ldots) = f^{\mathcal{O}}(\ldots,z_i,z_j,\ldots)S(z_i,z_j)$$

ii) Periodicity:



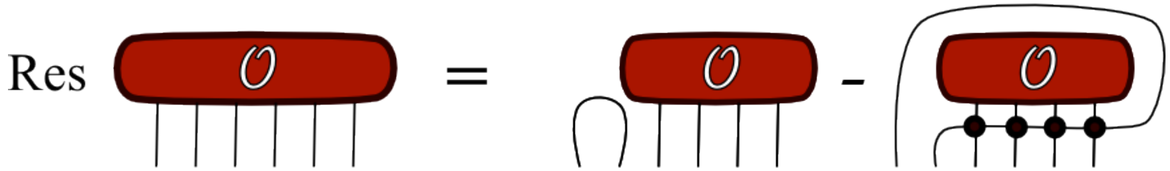
$$f^{\mathcal{O}}(z_1,z_2,\ldots,z_n) = f^{\mathcal{O}}(z_2,\ldots,z_n,z_1-2\omega_2)$$

Form factor Axioms

Define

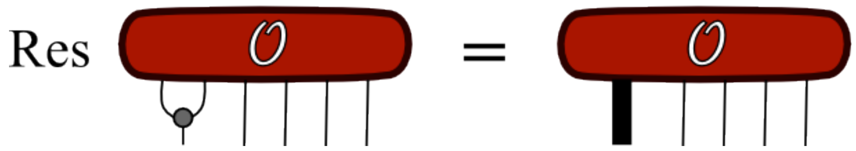
$$f^{\mathcal{O}}(z_1,\ldots,z_n) = \langle\Omega|\mathcal{O}|p(z_1),\ldots p(z_n)\rangle \ , \quad p(z_1) > \cdots > p(z_n)$$

iii) One-particle poles: for $\mathbf{p}_1(z_1) + \mathbf{p}_2(z_2) = 0$



$$\text{Res} \quad f^{\mathcal{O}}(z_1,\ldots,z_n) = 2iC_{12} f^{\mathcal{O}}(z_3,\ldots,z_n)(1 - S_{2n}\ldots S_{23})$$

iv) Bound state poles: for $\text{Res}_{z_1',z_2'} S_{12}(z_1,z_2) = R_{(12)}$



$$\text{Res} \ f^{\mathcal{O}}(z_1,z_2,\ldots,z_n) = \sqrt{2iR_{(12)}} f^{\mathcal{O}}(z_{12},z_3\ldots,z_n)$$

These “axioms” are obviously similar to relativistic case but are they correct? - Checked i)-iii) are satisfied by a number of perturbative checks ($\sqrt{\lambda} \gg 1$).

Can’t check iv) - existence of bound states - perturbatively but is apparent at “strong” coupling in spin-chain description.

Axioms are for local operators. Worldsheet symmetries are non-local and so decendant states satisfy modified axioms.

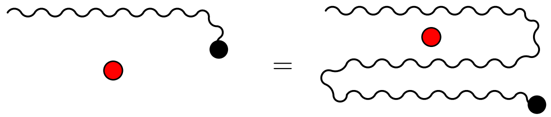
Worldsheet Symmetries

Due to gauge fixing the centally extended psu(2|2)^2 symmetries involve a non-locality

$$\mathbb{Q}_I = \int d\sigma \; \mathbb{J}_I \;, \quad \text{with} \quad \mathbb{J}_I = e^{i\epsilon_I X^-} \Omega_I$$

where $$X^- = -\frac{1}{g}\int_C d\sigma P_M \acute{X}^M + \text{fermions} \; .$$

This non-locality results in a non-trivial braiding of the currents with worldsheet fields



$$\mathbb{J}_{\hat{I}}(\sigma)\Phi_A(\sigma_0) = \hat{\Theta}_{\hat{I}}^{\hat{J}}[\Phi_A](\sigma_0)\mathbb{J}_{\hat{J}}(\sigma) \;, \quad \text{for} \quad \sigma > \sigma_0$$

which implies for the action of charges on products of fields

$$\hat{\mathbb{Q}}_{\hat{I}}[\Phi_{A_1}(\sigma_1)\Phi_{A_2}(\sigma_2)] = \hat{\mathbb{Q}}_{\hat{I}}[\Phi_{A_1}(\sigma_1)]\Phi_{A_2}(\sigma_2) + \hat{\Theta}_{\hat{I}}^{\hat{J}}[\Phi_{A_1}(\sigma_1)]\hat{\mathbb{Q}}_{\hat{J}}[\Phi_{A_2}(\sigma_2)] \; .$$

Worldsheet Hopf Algebra

This can be interpreted as a non-cocommutative coproduct (and in fact one can define an antipode and co-unit). Introducing a linear basis, e_a , for the universal enveloping algebra we can define the Hopf algebra as

$$e_a e_b = m_{ab}^c e_c \ , \quad \Delta(e_a) = \mu_a^{bc} e_b \otimes e_c$$

where the relation to the previous charges and braiding is

$$\mathbb{Q}_I = e_I \ , \quad \Theta_J^I = \mu_J^{aI} e_a$$

From the coproduct we can define the adjoint action of charges on fields

$$\mathrm{ad}_{e_a}(\Phi) = \mu_a^{bc} \ e_b \ \Phi \ s(e_c)$$

for example taking one of the psu(2|2) charges on a bosonic worldsheet field

$$\mathrm{ad}_{\mathbb{Q}_A{}^B}(X) = \mathbb{Q}_A{}^B X - e^{i\epsilon_{AB}\mathbb{P}} X e^{-i\epsilon_{AB}\mathbb{P}} \mathbb{Q}_A{}^B \ ,$$

Form Factor Ward Identity

Using the adjoint action we can derive the axioms for descendant operators from those for local operators. Consider

$$f^{\mathbb{Q}_A^B \mathcal{O}}(p(z_1), \ldots, p(z_n)) = \langle 0 | \mathrm{ad}_{\mathbb{Q}_A^B}(\mathcal{O}) | p_1, \ldots, p_n \rangle$$

and the known action of charges on asymptotic states

$$\mathbb{Q}_A{}^B | p_1, \ldots, p_n \rangle = \sum_{i=1}^{n} e^{i\epsilon_{AB} \sum_{j=1}^{i-1} p_j} (\mathbb{Q}_A{}^B)_i | p_1, \ldots, p_n \rangle$$

hence we find

$$\begin{aligned} f^{\mathbb{Q}_A^B \mathcal{O}}(z_1 + 2\omega_2, \ldots, z_n) &= e^{i\epsilon_{AB} p_1} f^{\mathbb{Q}_A^B \mathcal{O}}(z_2 \ldots, z_n, z_1) \\ &+ (1 - e^{i\epsilon_{AB} \sum_{i=1}^{n} p_i}) (\mathbb{Q}_A^B)_1 f^{\mathcal{O}}(z_2, \ldots, z_n, z_1) \end{aligned}$$

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when the total momentum is zero (natural from string theory p.o.v.). This modified axiom is very similar to recently proposed axioms for string vertex operator (Bajnok, Janik '15) where the local operator is replaced by a string vertex.

Exact form factors

We would like to find all order form factors by solving these axioms.

Ex. Consider two-particle form factors in relativistic theory. Answer can be written as

$$f^{\mathcal{O}}(\theta) = \mathcal{N}^{\mathcal{O}} K^{\mathcal{O}}(\theta) f_{\min}(\theta) \quad \text{s.t.} \qquad f_{\min}(\theta + 2\pi i) = S(-\theta) f_{\min}(\theta)$$

K is a even, periodic function which reproduces the pole structure and *N* is normalisation constant.

Remains to find a **minimal** solution i.e. one with no poles or zeros in the physical strip.

Worldsheet minimal solution

For the world-sheet theory, defined on a rapidity **torus** with periods $2\omega_1$ & $2\omega_2$, the minimal solution in a rank-one sector should satisfy:

$$f_{\min}(z_1 + 2\omega_2, z_2) = S(z_1, z_2) f_{\min}(z_1, z_2)$$

We will additionally assume that the form factors satisfy

$$f_{\min}(z_1 + \omega_2, z_2 + \omega_2) = f_{\min}(z_1, z_2)$$

which is essentially world-sheet parity (or analogue of CT transformation in relativistic model). Hence using $z_\pm = z_1 \pm z_2$ the equations we want to solve are

$$f_{\min}(z_+ + 2\omega_2, z_-) = f_{\min}(z_+, z_-)$$

$$f_{\min}(z_+, z_- + 2\omega_2) = S(z_+, z_-) f_{\min}(z_+, z_-)$$

An exact minimal solution

As in the relativistic case we can write the formal solution

$$\log f_{\min}(z_+, z_-) = \sum_{n=1}^{\infty} \log S(z_+, z_- - 2\omega_2 n)$$

and as in relativistic case we can perform the sum by Fourier transforming S-matrix along real axis

$$\log S(z_-, z_+) = \sum_{m=-\infty}^{\infty} h(z_+, m) e^{\frac{i\pi m z_-}{\omega_1}}$$

$$\Rightarrow \log f_{\min}(z_-, z_+) = \sum_{n=1}^{\infty} h(z_+, n) \cos\left(\frac{n\pi(z_- - \omega_2)}{\omega_1}\right) \operatorname{csch}\left(\frac{i\pi n \omega_2}{\omega_1}\right)$$

Not particularly useful answer, to be more concrete we can expand and compare perturbative answers.

Minimal MS-form factors

For the near-flat limit we can write the one-loop S-matrix in the appropriate integral form (helps to use the fact that the BDS part in near-flat limit is sG BB)

$$\ln S(\tilde\gamma,\theta) = -\frac{4\tilde\gamma}{\pi}\int_0^\infty dt\ \coth t \sinh\frac{t\theta}{i\pi} + \frac{2\tilde\gamma^2}{\pi^2}\int_0^\infty dt\ t\,(3-\tanh^2\frac{t}{2})\sinh\frac{t\theta}{i\pi} + \mathcal{O}(\tilde\gamma^3)\ .$$

which gives

$$f_{\rm min}(\theta) = 1 - \frac{\tilde\gamma}{\pi}(\theta - i\pi)\coth\frac{\theta}{2} + \frac{\tilde\gamma^2}{2\pi^2}(\theta - i\pi)^2\coth^2\frac{\theta}{2} + \frac{3\tilde\gamma^2}{4}{\rm csch}^2\frac{\theta}{2}$$
$$- \frac{\tilde\gamma^2}{\pi^2}(\theta - i\pi)\big[2 - (\theta - i\pi)\coth\theta\big]{\rm csch}\theta + \mathcal{O}(\tilde\gamma^3)$$

which correctly reproduces the discontinuities of the pert. form factor. The remaining “rational” terms are indeed even and periodic but don’t follow from a “minimality” condition for Y^2 operator and one needs to add higher derivative terms.

Similar for the near-BMN one-loop form factor in *a*=1/2 gauge.

Outlook

- We would like to solve the axioms and find exact multi-particle form factors.

- Can we use the form factors to compute all order structure constants?
    - Simplest will hopefully be quantum HHL states
        - See conjecture of [Bajnok, Janik and Wereszczyński 1404.4556]
    - Form factors as a step toward LCSFT for AdS/CFT space and so all three-point functions.
    - Such solutions would provide part of the string vertex operator (Janik/Bajnok).

- Would like to better understand the algebraic structure of the underlying symmetries (e.g. the quantum double).