Master Symmetry For Strings in $AdS_5 \times S^5$

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Abstract

In some field theories Master Symmetry [1] is an alternative to the monodromy, Backlund and BIZZ recursion approaches for demonstrating integrability and constructing an infinite tower of conserved charges. It also allows us to extend any solution of the equations of motion to a one-parameter family of solutions. Only very recently [2] has this symmetry been applied in the context of closed strings on $AdS_5 \times S^5$ and much is yet to be understood.

We aim to answer the following questions:

- How do the conserved charges relate to those of the Monodromy construction?
- How do we use the symmetry to construct explicit solutions of the string equations of motion?
- Are the new solutions physical?

Introduction & Overview

Let $\mathfrak{g} \in \mathrm{PSU}(2,2|4)$ and $A = -\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g}$ the associated Maurer-Cartan form which can be decomposed as $A = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}$ by the $\mathbb{Z}_4$ grading of the algebra. Then the string action is given by

$$S = \int STr\left(A^{(2)} \wedge *A^{(2)} - A^{(1)} \wedge A^{(3)}\right) \tag{1}$$

The model is integrable with Lax Connection $A(u)$ given by

$$A(u) = A^{(0)} + e^{-u}A^{(1)} + \cosh(2u)A^{(2)} + \sinh(2u)A^{(2)} + e^{u}A^{(3)} \tag{2}$$

We refer to $\mathfrak{g} \to \mathfrak{g}_u$ as master symmetry. We then suppose there exists a field $\mathfrak{g}_u$ such that

$$A(u) = -\mathfrak{g}_u^{-1}d\mathfrak{g}_u \tag{3}$$

We make the following ansatz for $\mathfrak{g}_u$;

$$\mathfrak{g}_u = \chi_u \mathfrak{g}_u \tag{4}$$

Expanding around $u = 0$, consistency then requires that

$$d\chi^{(n)} = -\sum_{k=0}^{n} \chi^{(k)} L^{(n-k)}, \quad L(u) := \mathfrak{g}(A(u) - A)\mathfrak{g}^{-1} \tag{5}$$

It follows that $j_{(n)} = -\sum_{k=0}^{n} \chi^{(k)} * L^{(n-k)}$ is a conserved current $\forall n \geq 1$.

The charge is then defined as

$$Q^{(n)} = \int_0^{2\pi} d\sigma j^{\tau}_{(n)} = -\int_0^{2\pi} d\sigma \sum_{k=0}^{n} \chi^{(k)} L_{\sigma}^{(n-k)} = \chi^{(n)}(2\pi) - \chi^{(n)}(0) \tag{6}$$

Master Symmetry Properties

- $Q^{(n)}$ are non-abelian and in general not conserved. In the decompactification limit the corresponding $Q^{(n)}$ are conserved.
- $Q^{(1)}$ is the Noether charge corresponding to global $\mathrm{PSU}(2,2|4)$ symmetry of the action, up to normalization.
- Since A and $A(u)$ satisfy the same equations of motion, we can promote any conserved current j to a one parameter family of conserved currents j_u by replacing $A \to A(u)$.
- Given a periodic solution $\mathfrak{g}$ of the equations of motion, $\mathfrak{g}_u$ is not periodic. In particular,

$$\mathfrak{g}_u(\sigma + 2\pi) - \mathfrak{g}_u(\sigma) = \left(\sum_{n=0}^{\infty} Q^{(n)}u^n\right)\mathfrak{g}(\sigma) \tag{7}$$

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Monodromy Vs Master Symmetry

Monodromy and Master Symmetry both provide a way of demonstrating integrability. The monodromy matrix in infinite volume is defined as

$$T(u) = \mathrm{Pexp}\left(\int_{-\infty}^{\infty} \mathrm{d}\sigma L_{\sigma}\right) \tag{8}$$

$T(u)$ is actually very closely related to the Master Symmetry charge generators:

$$\chi_u^{-1}(\infty) = T(u)\chi_u^{-1}(-\infty) \tag{9}$$

In the case of finite volume $\mathrm{Str}\,T(u)$ produces conserved quantities when expanded in u. Their constant values coincide with the master symmetry charges at a certain time $\tau = \tau_0$.

Explicit Bosonic String Solutions

We use the master symmetry to promote explicit string solutions to a one-parameter family of solutions. We can write

$$A = e^m P_m + \frac{1}{2}\omega^{mn} J_{mn} \tag{10}$$

where ω^{mn} and $e^m = e^m_M dX^M$ are the spin connection and vielbien respectively. Lifting the symmetry $A \to A_u$ of the action to the embedding coordinates corresponds to solving the following system of PDEs

$$e^m_M(u)\partial_\alpha X^M(u) = \cosh(2u)e^m_M\partial_\alpha X^M + \sinh(2u)\eta_{\alpha\beta}\epsilon^{\beta\rho}e^m_M\partial_\rho X^M \tag{11}$$

As an explicit example, we parametrize AdS_5 as the surface in $\mathbb{R}^{2,4}$ defined by

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -1 \tag{12}$$

and further parametrize

$$X_0 = Z_0 + iZ_5, \quad X_1 = Z_1 + iZ_2, \quad X_2 = Z_3 + iZ_4 \tag{13}$$

Then

$$X_0 = \sqrt{1+\rho^2}e^{it}, \quad X_1 = \rho\cos(\zeta)e^{i\psi_1}, \quad X_2 = \rho\sin(\zeta)e^{i\psi_2} \tag{14}$$

with

$$\begin{aligned} &\rho = \rho_0, \quad \zeta = \pi/4, \quad t = \kappa\tau, \\ &\psi_1 = w\tau + \sigma, \quad \psi_2 = w\tau - \sigma \end{aligned} \tag{15}$$

is a solution of the string equations of motion ($\kappa = 2\rho$, $w^2 = \rho^2 + 1$). Solving the PDEs we find that the following is also a solution

$$\begin{aligned} \rho(u) &= \rho_0, \quad \zeta(u) = \pi/4 \\ t(u) &= \kappa\sinh(2u)\sigma - \kappa\cosh(2u)\tau \\ \psi_1(u) &= \sigma(\cosh(2u) - w\sinh(2u)) + \tau(w\cosh(2u) - \sinh(2u)) \\ \psi_2(u) &= -\sigma(\cosh(2u) + w\sinh(2u)) + \tau(w\cosh(2u) + \sinh(2u)) \end{aligned} \tag{16}$$

Again we see that periodicity is broken in general.

Future Directions & Questions

- Despite huge similarities in the constructions, Master Symmetry is not a specific case of the Backlund transformation considered in [3]. How are the two approaches related?
- Does the master symmetry exist for the quantum string? If so it may be possible to construct an infinite tower of quantum conserved charges.
- Is there an analogue of master symmetry in $\mathcal{N} = 4$ Super Yang Mills?
- Can the master symmetry be extended to a symmetry of the η-deformed string?

References

[1] Thomas Klose, Florian Loebbert, Hagen Munkler "Nonlocal Symmetries, Spectral Parameter and Minimal Surfaces in AdS/CFT", arXiv:1610.01161

[2] Osvaldo Chandia, William Divine Linch III, Brenno Carlini Vallilo "Master symmetry in the $AdS_5 \times S^5$ pure spinor string", arXiv:1607.00391

[3] Gleb Arutyunov, Marija Zamaklar "Linking Backlund and Monodromy Charges for Strings on $AdS_5 \times S^5$", arXiv:0504144