Truncated Hilbert Space Approach for the 1+1D φ^4 theory

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Motivation

- QFTs almost never solvable exactly $\to$ approximation methods (perturbation&improvements, semiclassics, lattice, TCSA, etc.)
- **Hamiltonian truncation** on free massive Fock space used to be considered insufficient
- QFT in $1+1$ dimensions (Minkowski):
  - scalar theories super-renormalizable (finite after normal ordering)
  - integrable theories exist $\to$ integrable framework based on S-matrix bootstrap
- φ^4 Landau-Ginzburg model: very interesting (non-integrable) toy model
  - two phases, topological excitations (kink), bound states, resonances,...
  - **S-matrix similar to integrable theories below particle creation treshold**
- truncated space methods also well-suited to compute form factors

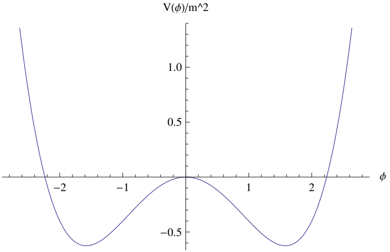
Recent works

- A. Coser, M. Beria, G. Brandino, R. Konik, G. Mussardo, *J. Stat. Mech.* **12** (2014) P12010 [arXiv:1409.1494]
- S. Rychkov, L. G. Vitale, *Phys. Rev.* **D 91** (2015) 085011 [arXiv:1412.3460]
- S. Rychkov, L. G. Vitale, *Phys. Rev.* **D 93** (2016) 065014 [arXiv:1512.00493]
- J. Elias-Miro, M. Montull, M. Riembau, *JHEP* **04** (2016) 144 [arxiv:1512.05746]
- **Z. Bajnok, M. Lajer, *Truncated Hilbert space approach to the 2d φ^4 theory*,** *JHEP* **10** (2016) 050 **[arXiv:1512.06901]**

Classical action

$$S = \int d^2x \mathscr{L}(x), \qquad \mathscr{L}(x) = \frac{1}{2}\partial_\mu \varphi(x) \partial^\mu \varphi(x) - V(\varphi(x))$$

$$V(\varphi) = m^2 \left(-\frac{1}{2}\varphi^2 + \frac{g}{6}\varphi^4 \right)$$



- $1+1$ dimensions, $\hbar = c = 1$
- Symmetries: Poincaré, internal $\mathbb{Z}_2$, spatial parity
- Ground state breaks $\mathbb{Z}_2$ parity

Fluctuations and kinks

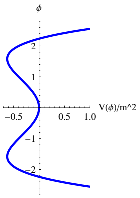
- Small classical fluctuations around minima:

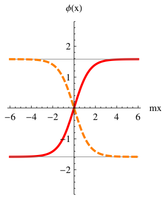
$$\omega_0 \approx \sqrt{\bar{m}_0^2 + k^2}, \qquad \bar{m}_0 = \sqrt{2}m$$

- Classical kink/antikink ($K/\overline{K}$)solution (stable):

$$\phi_\pm = \pm\sqrt{\frac{3}{2g}}\tanh\left(\frac{\bar{m}_0 x}{2}\right), \qquad M_0 = \frac{\bar{m}_0}{g}$$

- (anti)kink nonperturbative, disjunct topological sectors
- $K\overline{K}$ bound state (breather) classically not stable

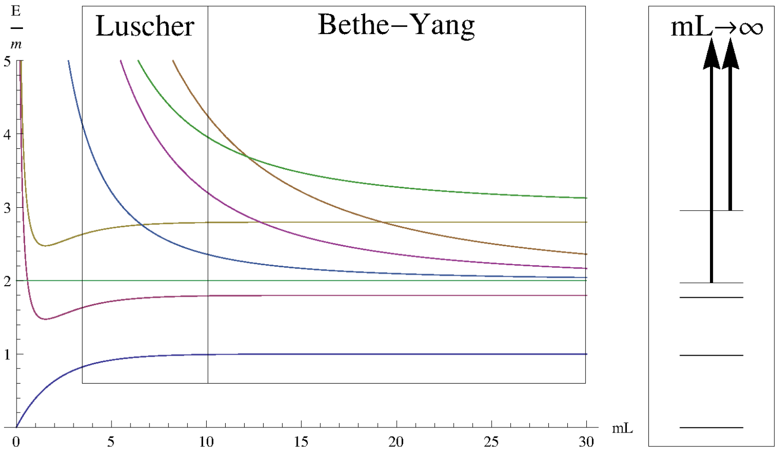




Quantum theory

- Information from **semiclassics**:
  - Quantum breathers $\longleftrightarrow$ elementary fluctuations around minima
  - New particle: excited kink K^*; mass corrections to K and K^*
  - Mussardo: at most two *stable* breathers in the spectrum
- **What we do**:
  - Finite volume L $\rightarrow$ momentum quantized
  - Canonical quantization $\rightarrow$ Hamilton operator H
  - Symmetries divide Hilbert Space $\mathscr{H}$ into disjoint sectors: momentum P, particle number parity, space inversion
  - Truncate $\mathscr{H}$ space below energy E_{max}, diagonalize H numerically in $P = 0$ sector

Typical Finite Volume $P = 0$ Spectrum



Canonical Quantization

- Hamiltonian for **finite volume** L:

$$H = \int_0^L \mathscr{H}\, dx = H_0 + V,$$

$$\mathscr{H} = \frac{1}{2} : (\partial_t \varphi)^2 : + \frac{1}{2} : \left(\partial_x \varphi^2\right) : + \frac{1}{2} m_0^2 : \varphi^2 : - \frac{1}{2} \left(m^2 + m_0^2\right) : \varphi^2 : + \frac{g}{6} m^2 : \varphi^4 :$$

- Boundary conditions:

$$\varphi(x) = \pm \varphi(x+L)$$

- Quantization of free boson:

$$\varphi(x,t) = \sum_n \frac{1}{\sqrt{2\omega_n L}} \left(a_n e^{-i\omega_n t + ik_n x} + a_n^\dagger e^{i\omega_n t - ik_n x}\right)$$

$$[a_n, a_m] = \left[a_n^\dagger, a_m^\dagger\right] = 0 \quad ; \qquad \left[a_n, a_m^\dagger\right] = \delta_{n,m}$$

$$\omega_n = \sqrt{m_0^2 + k_n^2} \quad ; \qquad k_n = \frac{2\pi}{L} n$$

- PBC: n integer, ABC: n half-integer

Normal Ordering

- Numerically convenient scheme: N.O. at mass scale m_0 and volume L:

$$\mathscr{H} = \mathscr{H}_0^{m_0,L} + g_0\,(m_0, L) + g_2\,(m_0, L) :\varphi^2:_{m_0,L} + g_4\,(m_0, L) :\varphi^4:_{m_0,L}$$

- Our definition of finite-volume model:

$$\mathscr{H} = \mathscr{H}_0^{\bar{m}_0,\infty} + g_0\,(\bar{m}_0, \infty) + g_2\,(\bar{m}_0, \infty) :\varphi^2:_{\bar{m}_0,\infty} + g_4\,(\bar{m}_0, \infty) :\varphi^4:_{\bar{m}_0,\infty}$$

- Projected operators connected to $::_{m_0,L}$ operators via Wick theorem
- Semiclassical compatibility: parameters are defined as:

$$g_0\,(\bar{m}_0, \infty) = 0, \qquad g_2\,(\bar{m}_0, \infty) = -\frac{3}{4}\bar{m}_0^2, \qquad g_4\,(\bar{m}_0, \infty) = \frac{g\,\bar{m}_0^2}{12}$$

Normal Ordering

- Free Hamiltonian:

$$H_0^{\mu,L} = \int_0^L :(\partial_t\varphi)^2 + (\partial_x\varphi)^2 + \mu^2\varphi^2: dx + E_0^{\pm}(\mu,L) = \sum_n \omega_n a_n^\dagger a_n + E_0^{\pm}$$

- Comparing renormalization schemes:

$$E_0^{\pm}(\mu,L) = \mu \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh\theta \log\left(1 \mp e^{-\mu L\cosh\theta}\right)$$

- Relate $::_{\mu,L}$ normal ordering to $::_{\mu',L'}$:

$$:\varphi^2:_{\mu',L'} = :\varphi^2:_{\mu,L} + \Delta Z$$
$$:\varphi^4:_{\mu',L'} = :\varphi^4:_{\mu,L} + 6\Delta Z :\varphi^2:_{\mu,L} + 3(\Delta Z)^2,$$

$$\Delta Z = Z(\mu,L) - Z(\mu',L'), \qquad Z(\mu,L) = \sum_{|n|<N} \frac{1}{2\omega_n L}$$

THSA

- We choose a free massive Fock basis at volume $mL_0 = 1$
- We truncate the $\mathscr{H}$-space at some energy cutoff E_{max} wrt to free theory
- dimensionless parameters:

$$\frac{H}{m} = h \quad ; \qquad mL = l \quad ; \quad \omega_n = m\tilde{\omega}_n = m\sqrt{1 + \frac{4\pi^2 n^2}{l^2}}$$

- Hamiltonian:

$$\tilde{h}(l) = \tilde{h}_0 + \int_0^L mdx \left[\tilde{g}_2(l) : \phi^2 : + \tilde{g}_4 : \phi^4 : + \tilde{g}_0(l)\right]$$

$$\tilde{h}_0(l) = \sum_n \tilde{\omega}_n(l) a_n^+ a_n + \tilde{e}_0^{\pm}(l), \qquad \tilde{e}_0^{\pm}(l) = E_0^{\pm}(mL)/m$$

- **2 dimensionless parameters**: g, l; m sets energy scale
- We solve $\tilde{h}(l)\psi = e(l)\psi$ numerically in truncated basis

Mini Hilbert Space

- Periodic boundary condition: "mini superspace"

$$\mathscr{H} = \mathscr{H}^{\rm mini} \otimes \tilde{\mathscr{H}}$$

- IR (zero momentum) mode has special role in low-lying spectrum

$$\phi(x,0) = \tilde{\phi}(x,0) + \phi_0 \quad ; \qquad \phi_0 = \frac{1}{\sqrt{2l}}(a_0 + a_0^+)$$

- Mini Hamiltonian: anharmonic oscillator (QM)

$$h^{\rm mini} = \tilde{\omega}_0\, a_0^+ a_0 + \frac{1}{2}\tilde{g}_2(l) :(a_0 + a_0^+)^2: + \frac{1}{4l}\tilde{g}_4 :(a_0 + a_0^+)^4:$$

$$h = h^{\rm mini} + \tilde{h}_0 + \int_0^L mdx \left[\tilde{g}_0(l) + \tilde{g}_2(l) :\tilde{\phi}^2: \right.$$
$$\left. + \tilde{g}_4 \left(:\tilde{\phi}^4: + 4 :\tilde{\phi}^3: \phi_0 + 6 :\tilde{\phi}^2: \phi_0^2\right)\right]$$

$$\tilde{h}_0 = h_0 - h_0^{\rm mini}; \qquad h_0^{\rm mini} = \tilde{\omega}_0\, a_0^+ a_0$$

See also: S. Rychkov, L. G. Vitale, "A Hamiltonian Truncation Study of the ϕ^4 Theory in Two Dimensions II: The Z_2-Broken Phase and the Chang Duality", arXiv:1512.00493 (2015)

Technical Implementation

- **periodic** BC:
  - generate basis at fix $mL_0 = 1$
  - compute lowest eigenpairs of $h_{mini}(L)$
  - obtain matrix elements of $H(L)$ as a direct product; use "exact" eigenstates in the minimal basis
  - compute lowest eigenpairs of $H(L)$
  - perform this to various energy truncations and volumes
- **antiperiodic** BC:
  - generate basis at fix $mL_0 = 1$
  - obtain matrix elements of $H(L) \to$ eigenpairs

Technical Implementation

- $mL_0 = 1$: the spectrum is close to the *conformal spectrum*, energy approx. measurable in $\frac{2\pi}{mL}$ units
- mode $(a_n^+)^{j_n}$ contributes to this "conformal energy" as $j_n|n|$.
- **Energy cutoffs** introduced as integer multiples of this unit, taken between 14 and 26 (oscillator content remains same)
- **Volume dependence** measured from $l = 1$ to $l = 30$, usually $l \in \mathbb{N}$
- minimal space contains 2000 basis vectors. We keep the lowest 6 "exact" minimal eigenstates
- symmetries help: we work in $P = 0$ sector. Even/odd particle number sectors treated separately. Currently not utilising space parity.

Extrapolation in Energy Cutoff

- Rychkov et al.: leading correction to H due to truncation:

$$\Delta H = -\frac{g^2}{e_{max}^2}\left\{\left(c_1 \log^2 e_{max} + c_2 \log e_{max} + c_3\right) V_0 + \left(c_4 \log e_{max} + c_5\right) V_2 + c_6 V_4\right\}$$

- where $V_n = \int_0^L :\varphi^n: dx$, $e_{max} = E_{max}/m$
- we extrapolate (fit) using the leading e_{cut} dependence:

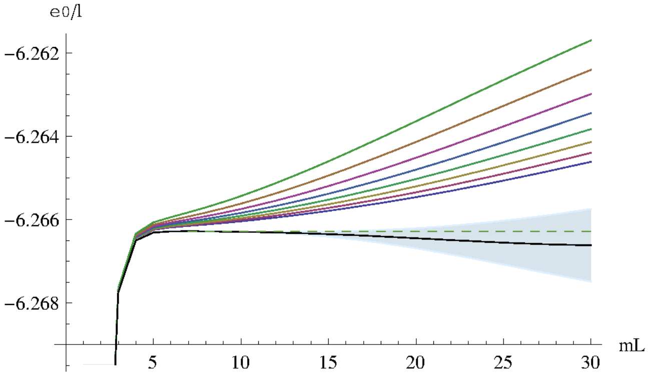
$$\frac{E(e_{cut})}{m} = a + b\frac{\log^2 e_{cut}}{e_{cut}^2} + c\frac{\log e_{cut}}{e_{cut}^2}$$

- obtanining an error:

$$\frac{E(e_{cut})}{m} = a + b\frac{\log^2 e_{cut}}{e_{cut}^2}$$

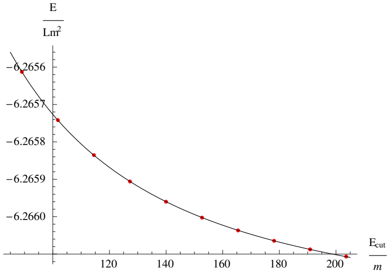
S. Rychkov, L. G. Vitale, "A Hamiltonian Truncation Study of the ϕ^4 Theory in Two Dimensions", *Phys. Rev.* **D91**, 085011 (2015)

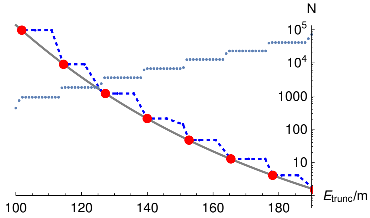
GS Energy Density Extrapolation



Coupling: $g = 0.06$. Energy density obtained at $l = 7$ is indicated with a dashed line.

GS Energy Density Extrapolation





Data points as a function of the truncation energy together with the fitted function for the ground-state energy density at $g = 0.06$ and volume $l = 10$.

Infinite volume data

- Parameters:
  - **masses**: $E_i = \sqrt{m_i^2 + p^2}$

    $$E_i(\theta) = m_i \cosh(\theta), \qquad p_i(\theta) = m_i \sinh(\theta)$$

  - **S-matrix**
    - only **elastic** processes below a kinematical treshold θ_t $\rightarrow$ integrable framework applicable (approximation)
    - two-particle S-matrices: e.g. S_{11}, S_{12}, S_{22}, S_{k1}, S_{k2}, $S_{k\bar{k}}$
    - general form restricted by unitarity, analicity, crossing, physical poles and branch cuts

2-particle S-matrices

- **General form**:

$$S(\theta) = S(\theta, i\alpha) \exp\left\{ -\int_{\theta_t}^{\infty} \frac{d\theta'}{2\pi i} \rho(\theta') \log S(\theta', \theta) \right\}$$

- θ_t: kinematic treshold rapidity difference for inelastic processes (e.g. $B_1 B_1 \to B_1 B_2$)
- **bound states** appear as poles, e.g. for $S_{11}(\theta)$:

$$S(\theta, i\alpha) = \frac{\sinh\theta + i\sin\alpha}{\sinh\theta - i\sin\alpha}$$

$$m_2(m_1, \alpha) = 2m_1 \cos\frac{\alpha}{2}$$

- Measuring $\rho(\theta)$ would be very interesting but difficult

Semiclassics

- Relevant quantities: masses and S-matrix
- Semiclassics: quantum fluctuations around cl. kink background
- first **SC correction** to kink mass:

$$M_{sc} = M_0 - m_0 c + \mathscr{O}(g), \qquad c = \frac{3}{2\pi} - \frac{1}{4\sqrt{3}}$$

- SC breathers (Mussardo, Riva, Sotkov):

$$m_n = 2M_{sc} \sin\left(n\frac{\zeta}{2}\right), \qquad \zeta = \frac{g}{1-gc} \approx g$$

- Stability against $B_n \to B_1 B_1$: $m_n < 2m_1$, SC: **always** true for m_2, **never** for m_3
- **Mussardo: there are at most two stable breathers for any coupling**

G. Mussardo, V. Riva, G. Sotkov, "Finite-Volume Form Factors in Semiclassical Approximation", Nucl. Phys. **B670** (2003) 464-478
G. Mussardo, "Neutral Bound States in Kink-like Theories", *Nucl.Phys.* **B779** (2007) 101-154

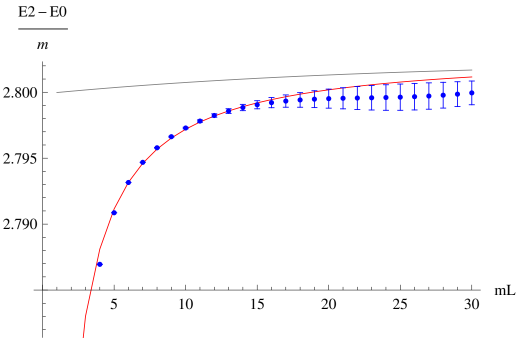
Finite size effects

- "Finite volume": space is compactified via $\varphi(x+L) = \pm\varphi(x)$
- Lüscher:

  finite volume spectrum of $H \Longleftrightarrow$ **infinite volume physical quantities**

- Leading order: "Bethe-Yang" corrections
  - **Polynomial** in L^{-1}
  - due to momentum quantization (BC)
- "Lüscher" corrections
  - **exponentially** small for large L
  - due to vacuum polarization (virtual particles)
- Finite volume vacuum splitting due to kink (**exponentially** small) $\rightarrow$ kink mass

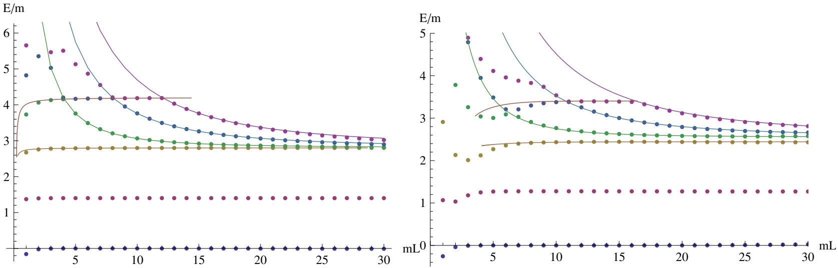
BY Description of B_2



Imaginary BY description of B_2 at $g = 0.06$ with leading finite size correction (gray). Fitting sensitive to α, here $\alpha = 0.06$.

$$imL \sinh \frac{\theta}{2} + \ln S_{11}(\theta) = 2\pi n, \quad n = 0$$

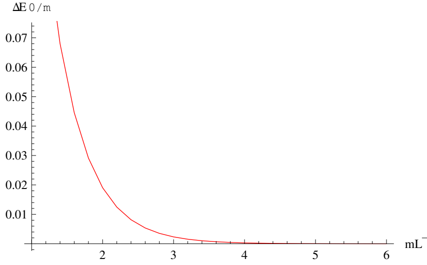
Low-lying spectra with BY lines

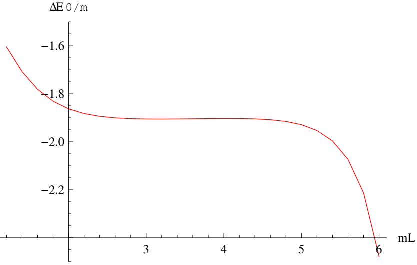


for $g = 0.06$ (left) and $g = 0.6$ (right)

$$e^{ip_j L} \prod_{n:n\neq j} S_{jn}(\theta_j - \theta_n) = \pm 1, \qquad E(L) = \sum_{n=1}^{N} m_n \cosh \theta_n(L)$$

Vacuum Splitting

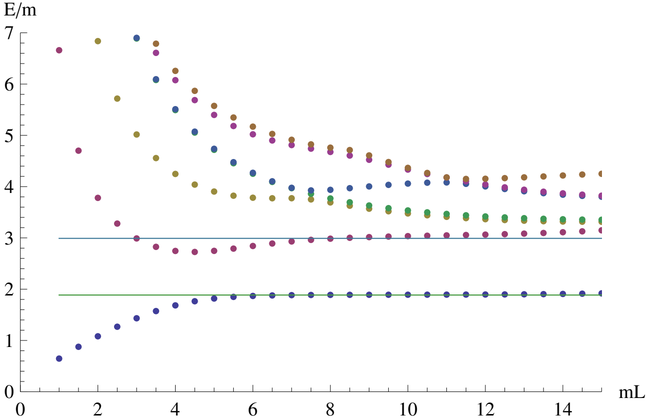




$$E_0^{odd}(L) - E_0^{even}(L) = \sqrt{\frac{2M}{\pi L}}e^{-ML} + \ldots$$

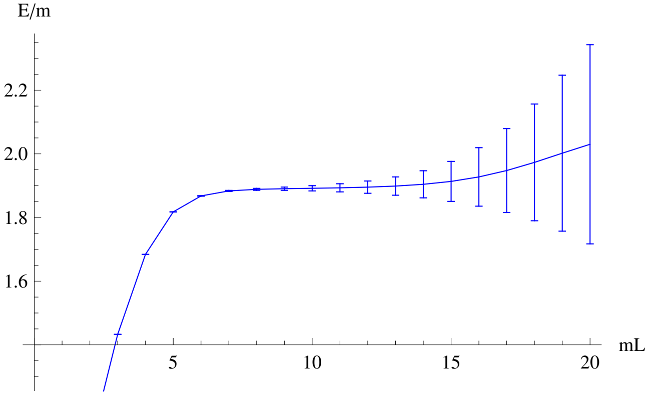
S. Rychkov, L. G. Vitale, "A Hamiltonian Truncation Study of the ϕ^4 Theory in Two Dimensions II: The Z_2-Broken Phase and the Chang Duality", arXiv:1512.00493 (2015)

Low-lying AP sector



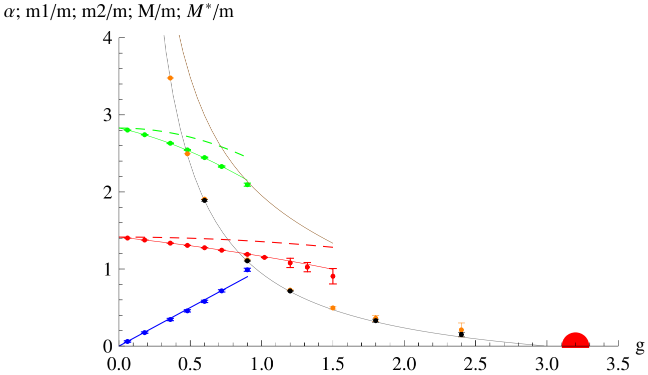
at $g = 0.6$. Resonance seems 3-particle involving a steady kink and two B_1s

Lowest Antiperiodic State



after substracting periodic vacuum. $g = 0.6$

Coupling dependence of masses



Conclusions

- Investigated $1+1D$ scalar φ^4 in symmetry breaking case
- numerical side: finite volume, THSA, periodic/antiperiodic BC, minimal space in periodic sector
- analytical side: 2-particle S-matrices, $L \to \infty$ quantities from finite size effects
- at most two neutral excitations: Mussardo's conjecture confirmed
- we see B_1, B_2 disappear from the spectrum at larger couplings, observe (hints of) phase transition
- two methods to obtain kink mass: vacuum splitting, antiperiodic lowest state

Thank you for your attention

Fock basis

- Elements eigenvectors of H_0:

$$\left|n_0, n_1, n_{-1}, \dots, n_k, \dots\right\rangle, \qquad n_i \in \mathbb{N}$$

- Orthonormality:

$$\left\langle n_0, \dots, n_k, n_{-k}, \cdots \,\middle|\, n'_0, \dots, n'_k, n'_{-k}, \dots\right\rangle = \prod_{i \in \mathbb{Z}} \delta_{n_i, n'_i}$$

- Ladder operators:

$$a_k \left|n_0, \dots, n_k, \dots\right\rangle = \begin{cases} \sqrt{n_k}\left|n_0, \dots, n_k - 1, \dots\right\rangle & , \quad n_k > 0 \\ 0 & , \quad n_k = 0 \end{cases}$$

$$a_k^\dagger \left|n_0, \dots, n_k, \dots\right\rangle = \sqrt{n_k + 1}\left|n_0, \dots, n_k + 1, \dots\right\rangle$$

- free vacuum:

$$\left|0\right\rangle \equiv \left|0, 0, \dots\right\rangle, \quad a_k \left|0\right\rangle = 0 \, \forall k$$