Sample Exam

Policy

Credit will be given for the best three out of four. All problems have equal weight.

Problem 1 (SI)

(i) Show that Maxwell's equations in vacuum imply that the $\mathbf{E}$ and $\mathbf{B}$ fields satisfy the wave equation.
Soln: The Maxwell equations in vacuum are:

$$\nabla \cdot \mathbf{E} = 0 \,, \qquad \nabla \cdot \mathbf{B} = 0 \,, \tag{1}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \,, \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0 \,. \tag{2}$$

Taking the time derivative of the first equation on the second line we find

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \tag{3}$$

which implies that

$$\nabla \times (c^2 \nabla \times \mathbf{B}) + \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \,. \tag{4}$$

We now use the identity

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \tag{5}$$

which can be derived in component form using the identity $\epsilon_{kij}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$ so that

$$\Big[\nabla \times (\nabla \times \mathbf{B})\Big]_i = \epsilon_{kin}\epsilon_{klm}\partial_j \partial_l B_m \tag{6}$$
$$= \partial_i \partial_m B_m - \partial_j \partial_j B_i \tag{7}$$
$$= \Big[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\Big]_i \,. \tag{8}$$

Using $\nabla \cdot \mathbf{B} = 0$ we have

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \,. \tag{9}$$

To find the corresponding equation for the $\mathbf{E}$ field we start from

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{10}$$

which implies

$$-\nabla \times (\nabla \times \mathbf{E}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \,, \tag{11}$$

from which the result follows as above.

(ii) The angular momentum for electromagnetic fields in vacuum is given by

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3x \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \right] . \quad (12)$$

Assuming that the fields are localized in space, i.e. vanish at spatial infinity, show that

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3x \left[\mathbf{E} \times \mathbf{A} + \sum_{j=1}^{3} E_j (\mathbf{x} \times \nabla) A_j \right] \quad (13)$$

where $\mathbf{A}$ is the usual vector potential.
Soln: Consider the i-th component of the vector $\mathbf{L}$ and, using the relation $\mathbf{B} = \nabla \times \mathbf{A}$, the integral

$$\left[\int d^3x \ \{ \mathbf{x} \times (\mathbf{E} \times (\nabla \times \mathbf{A}) \} \right]_i = \int d^3x \ \epsilon_{ijk} \epsilon_{klm} \epsilon_{mrs} x_j E_l \partial_r A_s \quad (14)$$

$$= \int d^3x \ (\epsilon_{ijr} x_j E_l \partial_r A_l - \epsilon_{ijs} x_j E_r \partial_r A_s) \quad (15)$$

$$(16)$$

where we use the ϵ tensor identity above. Integrating the second term by parts, dropping the boundary term, using $\partial_r E_r = 0$ and rearranging we find

$$\int d^3x \left[\epsilon_{irs} E_r A_s - E_l (\epsilon_{ijr} x_j \partial_r) A_l \right] = \int d^3x \left[\mathbf{E} \times \mathbf{A} + \sum_j E_j (\mathbf{x} \times \nabla) A_j \right]_i \quad (17)$$

as required.

(iii) Consider a monochromatic plane wave moving along the z-axis:

$$\mathbf{E} = \mathrm{Re} \left\{ \mathbf{E}_0 e^{ikz - i\omega t} \right\} \quad (18)$$

with

$$\mathbf{E}_0 = (E_{0x} \hat{x} + E_{0y} \hat{y}) \ . \quad (19)$$

Find the direction and magnitude of the polarisation ellipse (i.e. the semi-axis and the tilt angle).
Soln: Consider writing the polarisation vector as

$$\mathbf{E}_0 = \mathbf{b} e^{i\alpha} \quad (20)$$

where $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2$ with $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ and $\mathbf{b}_1 \perp \mathbf{b}_2$. This is completely general as it is only the relative phase of the two components that is physical (see fig.). If we choose our axis such that $\mathbf{b}_1 = b_1 \hat{\mathbf{x}}'$ and $\mathbf{b}_2 = b_2 \hat{\mathbf{y}}'$ then

$$E_{x'} = b_1 \cos(kz - \omega t - \alpha) \quad (21)$$

$$E_{y'} = -b_2 \sin(kz - \omega t - \alpha) \quad (22)$$

and

$$\frac{E_{x'}^2}{b_1^2} + \frac{E_{y'}^2}{b_2^2} = 1 \quad (23)$$

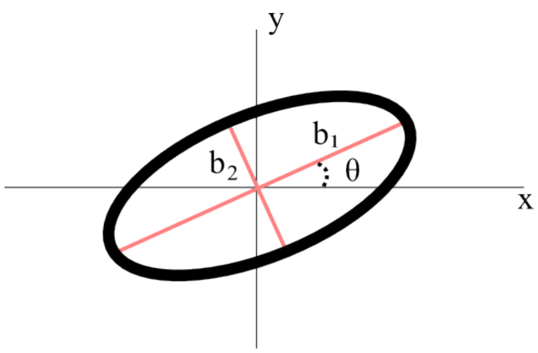

Figure 1: Polarization Ellipse

i.e. b_1 and b_2 describe the semi-major and semi-minor axis. The tilt angle is the angle between $\mathbf{b}_1$ and the x-axis.
If we write

$$E_{0x} = Ae^{i\delta_1} \ , \quad E_{0y} = Be^{i\delta_2} \tag{24}$$

then

$$\mathbf{E}_0 \cdot \mathbf{E}_0^* = b_1^2 + b_2^2 = A^2 + B^2 \tag{25}$$

and

$$\mathbf{E}_0 \times \mathbf{E}_0^* = -2iAB\sin\delta = -2i\mathbf{b_1} \times \mathbf{b_2} = -2ib_1b_2 \tag{26}$$

where $\delta = \delta_1 - \delta_2$. Hence we can solve for say b_2

$$b_1 = \frac{AB\sin\delta}{b_2} \Rightarrow A^2B^2\sin^2\delta + b_2^4 = b_2^2(A^2+B^2) \tag{27}$$

and so

$$b_2^2 = \frac{1}{2}((A^2+B^2) \pm \sqrt{(A^2+B^2)^2 - 4A^2B^2\sin^2\delta}) \tag{28}$$

where we take the $-$ sign for b_2 and the solution with the $+$ sign for the b_1. This gives the semi-minor and semi-major axis of the ellipse.
Now we want to find the tilt angle. To this end let us consider

$$\mathrm{Re}\left\{(\mathbf{E}_0 \cdot \mathbf{b}_1)(\mathbf{E}_0^* \cdot \mathbf{b}_2)\right\} = 0 \ . \tag{29}$$

This can be seen by plugging in the expression for $\mathbf{E}_0$ in terms of $\mathbf{b}$ and α.

$$\mathrm{Re}\left\{(\mathbf{E}_0 \cdot \mathbf{b}_1)(\mathbf{E}_0^* \cdot \mathbf{b}_2)\right\} = \mathrm{Re}\left\{\mathrm{i}\right\} \tag{30}$$

We can now write this as

$$\mathrm{Re}\left\{(\mathbf{E}_{0\mathrm{x}}\hat{\mathbf{x}} \cdot \mathbf{b}_1 + \mathbf{E}_{0\mathrm{y}}\hat{\mathbf{y}} \cdot \mathbf{b}_1)(\mathbf{E}_{0\mathrm{x}}^*\hat{\mathbf{x}} \cdot \mathbf{b}_2 + \mathbf{E}_{0\mathrm{y}}^*\hat{\mathbf{y}} \cdot \mathbf{b}_2)\right\} \tag{31}$$

$$= \mathrm{Re}\left\{\mathrm{A}^2\hat{\mathbf{x}} \cdot \mathbf{b}_1\hat{\mathbf{x}} \cdot \mathbf{b}_2 + \mathrm{B}^2\hat{\mathbf{y}} \cdot \mathbf{b}_1\hat{\mathbf{y}} \cdot \mathbf{b}_2) + \mathrm{ABe}^{-\mathrm{i}\delta}\hat{\mathbf{x}} \cdot \mathbf{b}_1\hat{\mathbf{y}} \cdot \mathbf{b}_2\right\} \tag{32}$$

Now using the facts that $\hat{\mathbf{x}} \cdot \mathbf{b}_1 = b_1\cos\theta$, $\hat{\mathbf{x}} \cdot \mathbf{b}_2 = -b_2\sin\theta$, $\hat{\mathbf{y}} \cdot \mathbf{b}_1 = b_1\sin\theta$, $\hat{\mathbf{y}} \cdot \mathbf{b}_2 = b_2\cos\theta$ we have

$$0 = -A^2b_1b_2\cos\theta\sin\theta + B^2b_1b_2\cos\theta\sin\theta - ABb_1b_2e^{i\delta}\sin^2\theta + ABb_1b_2e^{-i\delta}\cos^2\theta \tag{33}$$

and hence

$$\tan 2\theta = \frac{2AB\cos\delta}{A^2 - B^2} \ . \tag{34}$$

Problem 2 (SI)

(i) Find the differential equation satisfied by the Green function $G(\mathbf{x},t;\mathbf{x}',t')$ that gives

$$\psi = \int d^3\mathbf{x}'dt'\ G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t') \tag{35}$$

as a solution to

$$\Box\psi(\mathbf{x},t) = -4\pi f(\mathbf{x},t)\ . \tag{36}$$

Soln: If we consider

$$\Box\psi = \int d^3x'dt'\Box_{(\mathbf{x},t)}G(\mathbf{x},t;\mathbf{x}',t')f(\mathbf{x}',t') \tag{37}$$

this will be equal to $-4\pi f((\mathbf{x},t)$ if

$$\Box_{(\mathbf{x},t)}G(\mathbf{x},t;\mathbf{x}',t') = -4\pi\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta(t)\ . \tag{38}$$

(ii) If we assume that the Green function is only a function of $\mathbf{r} = \mathbf{x}-\mathbf{x}'$ and $\tau = t-t'$ and that it vanishes for $\tau < 0$ i.e. the retarded Green function, show that it is given by

$$G_r(\mathbf{r},\tau) = \frac{c}{r}\delta(r-c\tau)\ . \tag{39}$$

Soln: We want to find a solution to

$$\Box_{(\mathbf{r},\tau)}G(\mathbf{r},\tau) = -4\pi\delta^{(3)}(\mathbf{r})\delta(\tau)\ . \tag{40}$$

We perform a Fourier transform in all the spatial and the time direction

$$G(\mathbf{r},\tau) = \int d^3kd\omega\ e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)}g(\mathbf{k},\omega) \tag{41}$$

and use

$$\delta^{(3)}(\mathbf{r})\delta(\tau) = \int \frac{d^3kd\omega}{(2\pi)^4}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)} \tag{42}$$

to find

$$\left(\frac{\omega^2}{c^2}-k^2\right)g = -\frac{4\pi}{(2\pi)^4} \Rightarrow g = -\frac{1}{4\pi^3}\frac{1}{\left(\frac{\omega^2}{c^2}-k^2\right)}\ . \tag{43}$$

Hence

$$G(\mathbf{r},\tau) = \frac{1}{4\pi^3}\int d^3kd\omega\ \frac{1}{\left(\frac{\omega^2}{c^2}-k^2\right)}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega\tau)}\ . \tag{44}$$

Due to the singularities along the integration contour (taken as the real line) we must specify how to compute the integral. This can be either done by moving the contour around the poles or, equivalently, shifting the poles off the real axis. Here we take the later approach and define

$$G(\mathbf{r},\tau) = \frac{c^2}{4\pi^3}\lim_{\epsilon\to 0^+}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{r}}\int d\omega\ \frac{e^{-i\omega\tau}}{((\omega+i\epsilon)^2-c^2k^2)}\ . \tag{45}$$

In this case the poles are now at $\omega = -i\epsilon \pm ck$. This choice of shifting the poles into the lower half of the complex ω plane corresponds to choosing the Green function to vanish for $\tau < 0$, i.e. we are computing the retarded Green function. We can close the contour in the lower half plane for $\tau > 0$ as $e^{-i\omega\tau}$ vanishes along a semi-circular arc with infinite radius in the lower half plane. The ω integral can thus be done by evaluating the residues of the poles (note the contour is clockwise and so gives an extra minus sign). Thus we have

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int d\omega\,\frac{e^{-i\omega\tau}}{2ck}\Big[\frac{1}{(\omega+i\epsilon)-ck}-\frac{1}{(\omega+i\epsilon)+ck}\Big] &= -\left(\frac{2\pi i}{2ck}\right)\Big[e^{-ick\tau}-e^{ick\tau}\Big] \\ &= -\left(\frac{\pi i}{2ck}\right)\sin ck\tau\,. \end{aligned} \tag{46}$$

Hence for the retarded Green function

$$G_r = \frac{c}{2\pi^2}\int \frac{k^2 dk d\Omega}{k} e^{i\mathbf{k}\cdot\mathbf{r}}\sin(ck\tau) \tag{47}$$

$$\begin{aligned} &= \frac{c}{\pi}\int_0^\infty dk\,\frac{e^{ikr}-e^{-ikr}}{ir}\sin(ckr) \\ &= \frac{c}{2\pi r}\int_{-\infty}^\infty \Big[e^{ik(r-c\tau)}-e^{ik(r+c\tau)}\Big] \\ &= \frac{c}{r}\Big[\delta(r-c\tau)-\delta(r+c\tau)\Big]\,. \end{aligned} \tag{48}$$

Now if we remember that we have assumed that $\tau > 0$, or put differently there is a Heaviside Θ-function in front of the above expression, we see that the second δ-function can never contribute hence

$$G_r(\mathbf{r},\tau) = \frac{c}{r}\delta(r-c\tau)\,, \tag{49}$$

as required.

(iii) Write down an expression for the scalar potential due to an arbitrary charge distribution. Expand the result to show that to first order in $|\mathbf{x}'|/r$ where $|\mathbf{x}| = r$ the electric dipole potential for arbitrary time variation is

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\Big[\frac{1}{r^2}\mathbf{n}\cdot\mathbf{p}_{\rm ret} + \frac{1}{cr}\mathbf{n}\cdot\frac{\partial \mathbf{p}_{\rm ret}}{\partial t}\Big]\,, \tag{50}$$

with $\mathbf{p}_{\rm ret} = \mathbf{p}(t' = t - r/c)$.
Soln: The general expression for the scalar field is

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{1}{R}[\rho(\mathbf{x}',t')]_{\rm ret} \tag{51}$$

with $\mathbf{R} = \mathbf{x} - \mathbf{x}'$. We can use the expansion

$$R \simeq r - \hat{n}\cdot\mathbf{x}' \tag{52}$$

and

$$t' = t - R/c \simeq t - r/c + \hat{n}\cdot\mathbf{x}'/c = t_{\rm ret} + \hat{n}\cdot\mathbf{x}'/c \tag{53}$$

so that

$$\rho(\mathbf{x}',t') = \rho(\mathbf{x}',t_{\rm ret}) + \frac{\hat{n}\cdot\mathbf{x}'}{c}\frac{\partial\rho_{\rm ret}}{\partial t}\,, \tag{54}$$

where we use the notation $\rho(\mathbf{x}', t = t_{\rm ret}) = \rho_{\rm ret}$. Hence we find

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{1}{r}\left(1+\frac{\hat{n}\cdot\mathbf{x}'}{r}\right)\Big[\rho_{\rm ret}+\frac{\hat{n}\cdot\mathbf{x}'}{c}\frac{\partial\rho_{\rm ret}}{\partial t}\Big]\\ &= \frac{1}{4\pi\epsilon_0}\Big[\int d^3x'\,\frac{\rho_{\rm ret}}{r}+\int d^3x'\frac{\hat{n}\cdot\mathbf{x}'}{r^2}\rho_{\rm ret}+\frac{1}{r}\int d^3x'\frac{\hat{n}\cdot\mathbf{x}'}{c}\frac{\partial\rho_{\rm ret}}{\partial t}\Big] \qquad (55)\end{aligned}$$

If we drop the leading term (this is the monopole term which doesn't contribute to the radiation potential) the dipole terms are

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\Big[\frac{\hat{n}\cdot\mathbf{p}_{\rm ret}}{r^2}+\frac{1}{cr}\hat{n}\cdot\frac{\partial\mathbf{p}_{\rm ret}}{\partial t}\Big] \qquad (56)$$

where

$$\mathbf{p}_{\rm ret} = \int d^3x'\,\mathbf{x}'\rho_{\rm ret}\,. \qquad (57)$$

Problem 3 (G)

A charged particle, e, follows a trajectory $r^\mu(\tau)$ parameterised by the invariant time τ with four-velocity V^μ. The retarded Green function is given by

$$D_r(x-x') = \frac{\theta(x_0-x_0')}{2\pi}\delta[(x-x')^2]\,. \qquad (58)$$

(i) Write down an expression for the charge's four-current and show that the electromagnetic field strength can be written as

$$F^{\alpha\beta} = \frac{e}{V\cdot(x-r)}\frac{d}{d\tau}\Big[\frac{(x-r)^\alpha V^\beta-(x-r)^\beta V^\alpha}{V\cdot(x-r)}\Big]\,. \qquad (59)$$

(ii) Show that in a particular frame where $(x-r)^\alpha = (R, R\mathbf{n})$ i.e. where the relative location of the charge e is given by

$$\mathbf{R} = \mathbf{x}-\mathbf{r}(\tau) = R\mathbf{n}\,,$$

and $V^\alpha = (\gamma c, \gamma c\boldsymbol{\beta})$ i.e. $c\,\boldsymbol{\beta} = -d\mathbf{R}/dt$ is the 3-velocity and the derivatives denoted by the dot are taken with respect to the coordinate time, t, that

$$V\cdot(x-r) = \gamma cR(1-\boldsymbol{\beta}\cdot\mathbf{n}) \qquad (60)$$

$$\frac{dV^\alpha}{d\tau} = (c\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, c\gamma^2\dot{\boldsymbol{\beta}}+c\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}) \qquad (61)$$

$$\frac{d\big[V\cdot(x-r)\big]}{d\tau} = -c^2+\frac{dV}{d\tau}\cdot(x-r)\,. \qquad (62)$$

(iii) Show that the radiative part of the magnetic field can be written in this particular frame as

$$\mathbf{B} = \frac{e}{c}\Big[\frac{\mathbf{n}\times\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\Big]_{\rm ret}\,. \qquad (63)$$

Soln:

(i) The four current is given by

$$J^\alpha(x') = ec\int d\tau V^\alpha(\tau)\delta^{(4)}[x' - r(\tau)] \tag{64}$$

so that the gauge potential is given by

$$A^\alpha(x) = \frac{4\pi}{c}\int d^4x'\; D_r(x-x')J^\alpha(x')$$

$$\Rightarrow \quad A^\alpha(x) = 2e\int d^4x'\; V^\alpha(x')\Theta(x_0 - r_0(\tau))\delta[(x-r(\tau))^2]\,. \tag{65}$$

One can perform the integration by using

$$\delta[(x-r(\tau))^2] = \delta(f(\tau)) = \sum_{\tau_i^*}\frac{1}{\left|\frac{df}{d\tau}\right|_{\tau=\tau_i^*}}\delta(\tau-\tau_i^*) \tag{66}$$

where the sum is over the zeros of the functions $f(\tau_i^*) = 0$. The two solutions correspond to $x_0 = r_0 \pm |\mathbf{x}-\mathbf{r}|$, however because of the Θ-function only one solution contributes and we use

$$\frac{df}{d\tau} = -2V^\alpha(x-r)_\alpha\,. \tag{67}$$

Hence one can show that

$$A^\alpha(x) = \left.\frac{eV^\alpha(\tau)}{V\cdot(x-r(\tau))}\right|_{\tau=\tau_0}$$

where $(x-r(\tau_0))^2 = 0$ and $x_0 > r_0(\tau_0)$. To find $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ it is more efficient to start with

$$\partial^\alpha A^\beta = 2e\int d\tau\; V^\beta(\tau)\Big[\partial^\alpha\Theta(x_0 - r_0(\tau))\delta[(x-r(\tau))^2] + \Theta(x_0-r_0(\tau))\partial^\alpha\delta[(x-r(\tau))^2]\Big]\,.$$

The term $\partial^\alpha\Theta$ gives rise to an additional delta-function which results in this term only having support on the world-line of the radiating charge, this term we exclude by hand from our considerations. The second term we can evaluate with the aid of

$$\partial^\alpha\delta(f(\tau)) = \partial^\alpha f.\frac{d\tau}{df}.\frac{d\delta(f)}{d\tau} \tag{68}$$

where $f = [x-r(\tau)]^2$ so that $\partial^\alpha f = 2(x^\alpha - r^\alpha)$ and

$$\frac{d\tau}{df} = \frac{1}{\frac{df}{d\tau}} = [-2\dot{r}^\alpha(x-r)_\alpha]^{-1} \tag{69}$$

so that

$$\partial^\alpha A^\beta = -2e\int d\tau\; V^\beta(\tau)\frac{(x-r)^\alpha}{V\cdot(x-r)}\Theta(x_0-r_0(\tau))\frac{d}{d\tau}\delta((x-r(\tau))^2)\,. \tag{70}$$

We can now integrate by parts and neglecting boundary terms and derivatives of the Θ-function we find that

$$\partial^\alpha A^\beta = 2e\int d\tau\;\frac{d}{d\tau}\Big[\frac{(x-r)^\alpha V^\beta(\tau)}{V\cdot(x-r)}\Big]\Theta(x_0-r_0(\tau))\delta[(x-r(\tau)^2]\,. \tag{71}$$

This integral is again of the form

$$\int d\tau\, g(\tau)\Theta(x_0 - r_0(\tau))\delta[(x - r(\tau))^2] \tag{72}$$

and can be done as above by rewriting the delta-function. This gives

$$\partial^\alpha A^\beta = \frac{e}{V\cdot(x-r)}\frac{d}{d\tau}\left[\frac{(x-r)^\alpha V^\beta}{V\cdot(x-r)}\right]_{\tau=\tau_0} \tag{73}$$

where τ_0 is as above and so

$$F^{\alpha\beta} = \frac{e}{V\cdot(x-r)}\frac{d}{d\tau}\left[\frac{(x-r)^\alpha V^\beta - (x-r)^\beta V^\alpha}{V\cdot(x-r)}\right]_{\tau=\tau_0}, \tag{74}$$

with $(x - r(\tau_0))^2 = 0$ and $x_0 > r_0(\tau_0)$.

(ii) In the part we consider a specific frame and make use of the parametrisation

$$(x-r)^\alpha = (R, R\mathbf{n}), \quad V^\alpha = (\gamma c, \gamma c\boldsymbol{\beta}).$$

The first identity follows from the definition of the scalar product and the fact that the metric is given by $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$.

$$\begin{aligned} V\cdot(x-r) &= \eta_{\alpha\beta}V^\alpha(x-r)^\beta = c\gamma R - Rc\gamma\mathbf{n}\cdot\boldsymbol{\beta} \\ &= c\gamma R(1-\mathbf{n}\cdot\boldsymbol{\beta}). \end{aligned} \tag{75}$$

Next we take the derivative with respect to the invariant time but note that γ which depends on $\beta = |\boldsymbol{\beta}|$ is not a constant so that

$$\frac{dV^\alpha}{d\tau} = \left[c\frac{d\gamma}{d\tau}, c\frac{d(\gamma\boldsymbol{\beta})}{d\tau}\right]. \tag{76}$$

Now using the relation between the coordinate time and the invariant time

$$d\tau = \frac{dt}{\gamma} \tag{77}$$

and

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-\beta^2}} = \frac{\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}}{(1-\beta^2)^{3/2}} = \gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} \tag{78}$$

we have

$$\frac{dV^\alpha}{d\tau} = \left[c\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}), c\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + c\gamma^2\dot{\boldsymbol{\beta}}\right]. \tag{79}$$

Finally we can write

$$\frac{d(V\cdot(x-r))}{d\tau} = -V^2 + \frac{dV}{d\tau}\cdot(x-r) \tag{80}$$

and using $V^2 = c^2$ we find the required result.

(iii) As we want to find only the radiation terms we need only consider those terms which involve derivatives of V^α. Starting from the definition of the B_i component in terms of $F^{\alpha\beta}$ we have

$$\begin{aligned}
B_i^{\text{rad}} &= -\frac{1}{2}\epsilon_{ijk}F^{jk}_{\text{rad}} \\
&= \frac{e\epsilon_{ijk}}{2(V\cdot(x-r))^3}\Big[((x-r)^j\frac{dV^k}{d\tau}-(x-r)^k\frac{dV^j}{d\tau})((x-r)\cdot V) \\
&\quad -((x-r)^jV^k-(x-r)^kV^j)(x-r)\cdot\frac{dV}{d\tau}\Big] \\
&= \frac{-e\epsilon_{ijk}}{(\gamma cR(1-\boldsymbol{\beta}\cdot\mathbf{n}))^3}\Big[Rn^j(c\gamma^2\dot\beta^k+c\gamma^4\beta^k(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})(\gamma cR(1-\boldsymbol{\beta}\cdot\mathbf{n})) \\
&\quad -(Rn^j\gamma c\beta^k)(Rc\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}-Rc\gamma^2\mathbf{n}\cdot\dot{\boldsymbol{\beta}}-Rc\gamma^4\mathbf{n}\cdot\beta\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\Big] \\
&= \frac{-e\epsilon_{ijk}}{cR(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}n^j\left[\dot\beta^k+\beta^k\mathbf{n}\cdot\dot{\boldsymbol{\beta}}-\dot\beta^k\mathbf{n}\cdot\boldsymbol{\beta}\right] \qquad (81)
\end{aligned}$$

this can be written as

$$\mathbf{B}^{\text{rad}} = \frac{e}{cR(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\left[\dot{\boldsymbol{\beta}}\times\mathbf{n}+(\mathbf{n}\times\dot{\boldsymbol{\beta}})(\mathbf{n}\cdot\boldsymbol{\beta})-(\mathbf{n}\times\beta)(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\right]. \qquad (82)$$

Note that

$$\begin{aligned}
\mathbf{n}\times(\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}) &= \mathbf{n}\times(\mathbf{n}\times\{\mathbf{n}\times\dot{\boldsymbol{\beta}}\})-\mathbf{n}\times(\mathbf{n}\times\{\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\}) \\
&= -\mathbf{n}\times\dot{\boldsymbol{\beta}}-\mathbf{n}\times((\mathbf{n}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}-(\mathbf{n}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}) \\
&= \dot{\boldsymbol{\beta}}\times\mathbf{n}+(\mathbf{n}\times\dot{\boldsymbol{\beta}})(\mathbf{n}\cdot\boldsymbol{\beta})-(\mathbf{n}\times\beta)(\mathbf{n}\cdot\dot{\boldsymbol{\beta}}) \qquad (83)
\end{aligned}$$

so that

$$\mathbf{B}^{\text{rad}} = \frac{e}{cR(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\left[\mathbf{n}\times(\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\})\right] \qquad (84)$$

as required and where all fields are naturally evaluated at the retarded time. This is exactly of the form

$$\mathbf{B}^{\text{rad}} = \mathbf{n}\times\mathbf{E}^{\text{rad}} \qquad (85)$$

where $\mathbf{E}^{\text{rad}}$ was calculated in class by essentially similar means.

Problem 4 (G)

(i) Show that the total power per unit solid angle radiated by a non-relativistic particle of charge e and acceleration $\mathbf{a}$ is

$$\frac{dP_{\text{NR}}}{d\Omega} = \frac{e^2}{4\pi c^3}|\mathbf{a}|^2\sin^2\Theta \qquad (86)$$

where Θ is the angle between $\mathbf{a}$ and the unit radial vector $\mathbf{n}$.

(ii) The Lorentz invariant generalization of Larmor's formula for the total power radiated by an non-relativistic charge is

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \qquad (87)$$

where m is the rest mass, τ is the proper time, and p^μ is the particles four-momentum. Show that this does correctly reduce to Larmor's result.

(iii) A relativistic particle moves past a fixed charge Ze along an approximately straight-line path at impact parameter b, and approximately constant speed v (but nonetheless non-zero acceleration). Show that the total energy radiated is

$$\Delta W = \frac{\pi Z^2 e^6}{4m^2c^4\beta b^3}\left(\gamma^2 + \frac{1}{3}\right) . \tag{88}$$

You may use the result:

$$\int_1^\infty \frac{1}{y^3}\left(1+\frac{A}{y^2}\right)\frac{dy}{\sqrt{y^2-1}} = \frac{1}{16}(4+3A)\pi . \tag{89}$$

Soln:

(i) In the non-relativistic approximation the fields (as found in Question 3 above) become

$$\mathbf{B}^{\rm rad} = \frac{e}{cR}\Big[\mathbf{n}\times(\mathbf{n}\times\{\mathbf{n}\times\dot{\boldsymbol{\beta}}\})\Big] , \quad \mathbf{E}^{\rm rad} = \frac{e}{cR}\Big[\mathbf{n}\times\{\mathbf{n}\times\dot{\boldsymbol{\beta}}\}\Big] . \tag{90}$$

The Poynting vector (in Gaussian units) is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}|\mathbf{E}^{\rm rad}|^2\mathbf{n} . \tag{91}$$

Hence the power radiated per unit solid angle is

$$\begin{aligned}\frac{dP}{d\Omega} &= R^2\mathbf{S}\cdot\mathbf{n} \\ &= \frac{c}{4\pi}|R\mathbf{E}^{\rm rad}|^2 \\ &= \frac{e^2}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2 .\end{aligned}$$

We denote the angle between $\dot{\boldsymbol{\beta}}$ and $\mathbf{n}$ as Θ and use $\dot{\boldsymbol{\beta}} = \mathbf{a}/c$ so that

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\mathbf{a}|^2\sin^2\Theta . \tag{92}$$

(ii) To find the expression for the total power radiated into all angles we integrate over all angles. In this case it no longer matters in what direction $\dot{\boldsymbol{\beta}}$ points and so we can choose it to be along the z-axis, so that $\Theta = \theta$ where θ is the usual polar coordinate. Using

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\sin^2\theta = \frac{8\pi}{3} \tag{93}$$

we find that the Larmor formula is

$$P = \frac{2e^2}{3c^3}|\mathbf{a}|^2 . \tag{94}$$

We now consider the relativistic version

$$P = -\frac{2e^2}{3m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \tag{95}$$

where

$$p^\mu = (\gamma mc, \gamma m\mathbf{v}) . \tag{96}$$

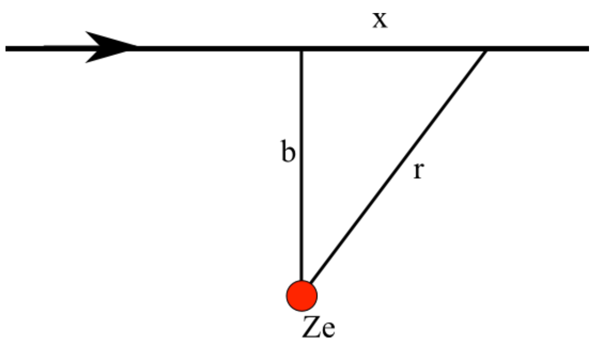

Figure 2: Particle trajectory

Using the fact that

$$\frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt} \tag{97}$$

and as $\gamma = (1-\beta^2)^{-1/2}$

$$\frac{d\gamma}{dt} = \gamma^3 \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} \tag{98}$$

we have

$$\frac{dp^\mu}{d\tau} = mc\gamma(\gamma^3 \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}, \gamma^3 \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) + \gamma \dot{\boldsymbol{\beta}}) \ . \tag{99}$$

Thus

$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = -m^2c^2\gamma^6((\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) + \gamma^{-2}(\dot{\boldsymbol{\beta}})^2) \ . \tag{100}$$

To take the non-relativistic limit we drop the $\boldsymbol{\beta}$ terms and set $\gamma \to 1$ and so

$$P = -\frac{2e^2}{3m^2c^3}(-m^2c^2(\dot{\boldsymbol{\beta}})^2) \tag{101}$$

which is the same as above. It is worthwhile to note that

$$(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) + \gamma^{-2}(\dot{\boldsymbol{\beta}})^2 = (\dot{\boldsymbol{\beta}})^2 - \beta^2(\dot{\boldsymbol{\beta}})^2 + (\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) = (\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2 \tag{102}$$

so that the relativistic formula can be written as

$$P = \frac{2e^2}{3c}\left[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2\right] \ . \tag{103}$$

(iii) In this problem we wish to calculate the power radiated by a relativistic particle which is in the field of target charge Ze. We assume that the particle is sufficiently far from the target and rapidly moving that we may treat its velocity as constant i.e. it moves along a straight line trajectory (Fig. 2), with $x = \sqrt{r^2 - b^2}$. To find the total energy radiated we integrate the power over all time

$$\Delta W = \int_{-\infty}^{\infty} dt \, P = 2\int_b^\infty \frac{dr}{dr/dt} P \tag{104}$$

where in the last step we change variables from the particle time t to it's radial position and we use the fact that it's motion is symmetric about it's point of closest approach. To calculate the power we need to find the rate of change of the particle momentum. The particle experiences a Coulomb force

$$\frac{d\mathbf{p}}{dt} = \frac{Ze^2}{r^2}\hat{r}\,. \tag{105}$$

Now we can write $p^\mu = (\mathcal{E}/c, \mathbf{p})$ where $\mathcal{E} = \gamma mc^2 = \sqrt{m^2c^4 + |\mathbf{p}|^2c^2}$ so that

$$-\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = \gamma^2\left(\frac{d\mathbf{p}}{dt}\right)^2 - \beta^2\gamma^2\left(\frac{d|\mathbf{p}|}{dt}\right)^2\,. \tag{106}$$

Hence

$$\left(\frac{d\mathbf{p}}{dt}\right)^2 = \frac{Z^2e^4}{r^4}\,, \quad \left(\frac{d|\mathbf{p}|}{dt}\right) = \frac{\mathbf{p}}{|\mathbf{p}|}\cdot\frac{d\mathbf{p}}{dt} = \frac{\mathbf{v}}{|\mathbf{v}|}\cdot\hat{r}\frac{Ze^2}{r^2}\,. \tag{107}$$

Plugging these expressions into the formula for the power radiated we have

$$\begin{aligned} P &= \frac{2e^2}{3m^2c^3}\Big[\gamma^2\frac{Z^2e^4}{r^4} - \beta^2\gamma^2\left(\frac{\mathbf{v}\cdot\hat{r}}{|\mathbf{v}|}\right)^2\frac{Z^2e^4}{r^4}\Big] \\ &= \frac{2Z^2e^6}{3m^2c^3}\frac{\gamma^2}{r^4}\Big[1 - (\boldsymbol{\beta}\cdot\hat{r})^2\Big]\,. \end{aligned}$$

Using simply trigonometry, from Fig. 2, we have

$$\boldsymbol{\beta}\cdot\hat{r} = \frac{v}{c}\sqrt{1 - \frac{b^2}{r^2}} \tag{108}$$

so that

$$P = \frac{2Z^2e^6}{3m^2c^3}\frac{1}{r^4}\left(1 + \frac{v^2b^2\gamma^2}{c^2r^2}\right)\,. \tag{109}$$

Thus we have for the energy radiated

$$\Delta W = 2\int_b^\infty P\frac{dr}{dr/dt} \tag{110}$$

where $r^2 = (vt)^2 + b^2$ (choosing the particle to be at its closest point at $t = 0$) so that

$$\frac{dr}{dt} = \frac{v}{r}\sqrt{r^2 - b^2} \tag{111}$$

which can simplified to

$$\Delta W = \left(\frac{4}{3}\frac{Z^2e^6}{m^2c^3v}\right)\int_b^\infty \frac{1}{r^3}\left(1 + \frac{v^2b^2\gamma^2}{c^2r^2}\right)\frac{dr}{\sqrt{r^2 - b^2}}\,. \tag{112}$$

After substituting $r = by$ and using the result given in the question, with $A = \frac{v^2}{c^2}\gamma^2$, we find

$$\begin{aligned} \Delta W &= \left(\frac{Z^2e^6\pi}{12m^2c^3vb^3}\right)(4 + 3\frac{v^2}{c^2}\gamma^2) \\ &= \left(\frac{Z^2e^6\pi}{4m^2c^4\beta b^3}\right)(\frac{1}{3} + \gamma^2) \end{aligned} \tag{113}$$

as required.