Sample Exam

Policy

Credit will be given for the best three out of four. All problems have equal weight.

Problem 1 (SI)

(i) Show that Maxwell's equations in vacuum imply that the $\mathbf{E}$ and $\mathbf{B}$ fields satisfy the wave equation.

(ii) The angular momentum for electromagnetic fields in vacuum is given by

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3x \left[\mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \right] . \tag{1}$$

Assuming that the fields are localized in space, i.e. vanish at spatial infinity, show that

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3x \left[\mathbf{E} \times \mathbf{A} + \sum_{j=1}^{3} E_j (\mathbf{x} \times \nabla) A_j \right] \tag{2}$$

where $\mathbf{A}$ is the usual vector potential.

(iii) Consider a monochromatic plane wave moving along the z-axis:

$$\mathbf{E} = \mathrm{Re} \left\{ \mathbf{E}_0 e^{ikz - i\omega t} \right\} \tag{3}$$

with

$$\mathbf{E}_0 = (E_{0x}\hat{x} + E_{0y}\hat{y}) . \tag{4}$$

Find the direction and magnitude of the polarisation ellipse (i.e. the semi-axis and the tilt angle).

Problem 2 (SI)

(i) Find the differential equation satisfied by the Green function $G(\mathbf{x}, t; \mathbf{x}', t')$ that gives

$$\psi = \int d^3\mathbf{x}' dt' \; G(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t') \tag{5}$$

as a solution to

$$= -4\pi f(\mathbf{x}, t) . \tag{6}$$

(ii) If we assume that the Green function is only a function of $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $\tau = t - t'$ and that it vanishes for $\tau < 0$ i.e. the retarded Green function, show that it is given by

$$G_r(\mathbf{r}, \tau) = \frac{c}{r} \delta(r - c\tau) . \tag{7}$$

(iii) Write down an expression for the scalar potential due to an arbitrary charge distribution. Expand the result to show that to first order in $|\mathbf{x}'|/r$ where $|\mathbf{x}| = r$ the electric dipole potential for arbitrary time variation is

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\mathbf{n}\cdot\mathbf{p}_{\text{ret}} + \frac{1}{cr}\mathbf{n}\cdot\frac{\partial\mathbf{p}_{\text{ret}}}{\partial t}\right], \tag{8}$$

with $\mathbf{p}_{\text{ret}} = \mathbf{p}(t' = t - r/c)$.

Problem 3 (G)

A charged particle, e, follows a trajectory $r^\mu(\tau)$ parameterised by the invariant time τ with four-velocity V^μ. The retarded Green function is given by

$$D_r(x - x') = \frac{\theta(x_0 - x_0')}{2\pi}\delta[(x - x')^2]\,. \tag{9}$$

(i) Write down an expression for the charge's four-current and show that the electromagnetic field strength can be written as

$$F^{\alpha\beta} = \frac{e}{V\cdot(x-r)}\frac{d}{d\tau}\left[\frac{(x-r)^\alpha V^\beta - (x-r)^\beta V^\alpha}{V\cdot(x-r)}\right]. \tag{10}$$

(ii) Show that in a particular frame where $(x - r)^\alpha = (R, R\mathbf{n})$ i.e. where the relative location of the charge e is given by

$$\mathbf{R} = \mathbf{x} - \mathbf{r}(\tau) = R\mathbf{n}\,,$$

and $V^\alpha = (\gamma c, \gamma c\boldsymbol{\beta})$ i.e. $c\,\boldsymbol{\beta} = -d\mathbf{R}/dt$ is the 3-velocity and the derivatives denoted by the dot are taken with respect to the coordinate time, t, that

$$V\cdot(x - r) = \gamma c R(1 - \boldsymbol{\beta}\cdot\mathbf{n}) \tag{11}$$

$$\frac{dV^\alpha}{d\tau} = (c\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}, c\gamma^2\dot{\boldsymbol{\beta}} + c\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta}) \tag{12}$$

$$\frac{d\big[V\cdot(x-r)\big]}{d\tau} = -c^2 + \frac{dV}{d\tau}\cdot(x - r)\,. \tag{13}$$

(iii) Show that the radiative part of the magnetic field can be written in this particular frame as

$$\mathbf{B} = \frac{e}{c}\left[\frac{\mathbf{n}\times\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{\text{ret}}. \tag{14}$$

Problem 4 (G)

(i) Show that the total power per unit solid angle radiated by a non-relativistic particle of charge e and acceleration $\mathbf{a}$ is

$$\frac{dP_{\text{NR}}}{d\Omega} = \frac{e^2}{4\pi c^3}|\mathbf{a}|^2\sin^2\Theta \tag{15}$$

where Θ is the angle between $\mathbf{a}$ and the unit radial vector $\mathbf{n}$.

(ii) The Lorentz invariant generalization of Larmor's formula for the total power radiated by an non-relativistic charge is

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}\right) \tag{16}$$

where m is the rest mass, τ is the proper time, and p^μ is the particles four-momentum. Show that this does correctly reduce to Larmor's result.

(iii) A relativistic particle moves past a fixed charge Ze along an approximately straight-line path at impact parameter b, and approximately constant speed v (but nonetheless non-zero acceleration). Show that the total energy radiated is

$$\Delta W = \frac{\pi Z^2 e^6}{4m^2c^4\beta b^3}\left(\gamma^2 + \frac{1}{3}\right) . \tag{17}$$

You may use the result:

$$\int_1^\infty \frac{1}{y^3}\left(1 + \frac{A}{y^2}\right)\frac{dy}{\sqrt{y^2-1}} = \frac{1}{16}(4+3A)\pi . \tag{18}$$