Exam Solutions

Problem 1

(i) As the boundary conditions depend only of θ the full problem has azimuthal symmetry. The general expression for the scalar potential with azimuthal symmetry is

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta) \ . \tag{1}$$

i.e. there is no ϕ dependence as the problem has azimuthal symmetry.

In the absence of additional charges or external fields at spatial infinity the potential must be finite i.e. we take $\Phi \to 0$ as $r \to \infty$, while at zero we similarly demand that the potential is finite i.e. $\Phi \to \text{constant as } r \to 0$.

(ii) At $r = a$ the scalar potential is given by

$$\Phi(a,\theta,\phi) = kE_0 a P_1(\cos\theta) \ . \tag{2}$$

Thus we need $A_0 = B_0 = 0$ and $A_1 a + B_1 a^{-2} = kE_0 a$ while $A_2 = B_2 = \cdots = 0$. Considering the region $r > a$ we want the potential to fall off at infinity i.e. $\Phi \to 0$ as $r \to \infty$. This implies $A_1 = 0$ and hence $B_1 = kE_0 a^3$ so

$$\Phi = kE_0 \frac{a^3}{r^2} \cos\theta \ , \quad r > a \ . \tag{3}$$

Considering the region $r < a$ we want the potential to be finite at the origin as there are no charges there, hence $B_1 = 0$ and so $A_1 = kE_0$. Thus

$$\Phi = kE_0 r \cos\theta \ . \tag{4}$$

(iii) As there are again no charges at the origin we have $B_\ell = 0$ for all ℓ and hence

$$\Phi(a,\theta,\phi) = \sum_{\ell=0}^{\infty} A_\ell a^\ell \ P_\ell(\cos\theta) \ . \tag{5}$$

We can find the A_ℓ coefficients by using orthogonality of the Legendre polynomials,

$$\begin{aligned}
\int_{-1}^{1} V(\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) &= \int_{-1}^{1} \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) \\
&= \sum_{\ell=0}^{\infty} A_\ell a^\ell \frac{2}{2\ell+1} \delta_{\ell\ell'} \\
&= A_{\ell'} a^{\ell'} \frac{2}{2\ell+1} \ .
\end{aligned} \tag{6}$$

That is

$$\begin{aligned}
A_\ell &= \frac{2\ell+1}{2a^\ell} \int_{-1}^{1} V(\theta) P_\ell(\cos\theta)\, d(\cos\theta) \\
&= \frac{2\ell+1}{2a^\ell} V\left[1-(-1)^\ell\right] \int_{0}^{1} P_\ell(\cos\theta)\, d(\cos\theta) \ .
\end{aligned} \tag{7}$$

This is zero for all even ℓ. For $\ell = 1$ we have

$$\int_0^1 x dx = \frac{1}{2} \tag{8}$$

and hence $A_1 = \frac{3}{2a}V$ so that

$$\Phi(r,\theta,\phi) = \frac{3}{2}V\left(\frac{r}{a}\right)P_1(\cos\theta) + \dots\ . \tag{9}$$

Problem 2

(i) The equations of motion are

$$n_i : \qquad \partial^2 n_i - \lambda n_i = 0 \tag{10}$$

where we use the short hand $\sum_{i=1}^3 n_i n_i = n^2$. The canonical stress-energy tensor is

$$T^{\mu\nu} = \partial^\mu n_i \partial^\nu n_i - g^{\mu\nu}\mathcal{L} \tag{11}$$

so that

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial^2 n_i \partial^\nu n_i + \partial^\mu n_i \partial^\mu \partial^\nu n_i - \partial^\nu(\tfrac{1}{2}\partial^\mu n_i \partial_\mu n_i + \tfrac{1}{2}\lambda(n_i n_i - 1)) \\
&= +\lambda n_i \partial^\nu n_i + \partial^\mu n_i \partial^\mu \partial^\nu n_i - (\partial^\nu \partial^\mu n_i)\partial_\mu n_i - \lambda(\partial^\nu n_i) n_i \\
&= 0\ .
\end{aligned} \tag{12}$$

(ii) The transformed Lagrangian is

$$\begin{aligned}
\mathcal{L}' &= \tfrac{1}{2}\sum_{i,j,k=1}^{3} \partial_\mu(M_{ij}n_j)\partial^\mu(M_{ik}n_k) + \tfrac{1}{2}\lambda(\sum_{i,j,k=1}^{3}(M_{ij}n_j)(M_{ik}n_k) - 1) &(13)\\
&= \tfrac{1}{2}\sum_{j,k=1}^{3}\left(\sum_{i=1}^{3} M_{ij}M_{ik}\right)\partial_\mu n_j \partial^\mu n_k + \tfrac{1}{2}\lambda(\sum_{j,k=1}^{3}(\sum_{i=1}^{n} M_{ij}M_{ik})n_j n_k - 1) &(14)
\end{aligned}$$

As the transformation is orthogonal we have $M^T M = 1$ or

$$\sum_{i=1}^{n} M_{ij}M_{ik} = \sum_{i=1}^{n} M^T_{ji}M_{ik} = \delta_{ji} \tag{15}$$

and using the Kronecker-delta's we have

$$\mathcal{L}' = \tfrac{1}{2}\sum_{j=1}^{3}\partial_\mu n_j \partial^\mu n_j + \tfrac{1}{2}\lambda(\sum_{j=1}^{3} n_j n_j - 1) \tag{16}$$

as required.

(iii) As λ is now a field we have an additional equation of motion:

$$n_i n_i = 1\ . \tag{17}$$

We wish to remove the dependence of λ from the n_i equations of motion and to this end we can multiply the equation of motion by n_i and sum over i. This implies that

$$\lambda = n_i \partial^2 n_i\ . \tag{18}$$

We can now differentiate $n^2 = 1$ twice to show that

$$(\partial_\mu n_i)n_i = 0 \Rightarrow (\partial^2 n_i)n_i = -(\partial_\mu n_i)(\partial^\mu n_i) \tag{19}$$

and so $\lambda = -(\partial_\mu n_i)(\partial^\mu n_i)$ as required.

Problem 3

(i) We can derive the equations of motion by using the Euler-Lagrange equations,

$$\frac{d}{d\tau}\Big[\frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)}\Big] - \frac{\partial L}{\partial x^\mu} = 0 \,. \tag{20}$$

Which gives us

$$\frac{d}{d\tau}\Big[-m\frac{dx_\mu}{d\tau} - \frac{q}{c}A_\mu\Big] + \frac{q}{c}\frac{dx^\nu}{d\tau}\frac{\partial A_\nu}{\partial x^\mu} = 0 \,. \tag{21}$$

So we have

$$\begin{aligned} & -m\frac{d^2x_\mu}{d\tau^2} - \frac{q}{c}(\partial_\nu A_\mu)\dot{x}^\nu + \frac{q}{c}(\partial_\mu A_\nu)\dot{x}^\nu = 0 \\ \Rightarrow \qquad & m\frac{d^2x_\mu}{d\tau^2} = -\frac{q}{c}F_{\nu\mu}\dot{x}^\nu \,, \quad \text{or} \quad \frac{du^\mu}{d\tau} = \frac{q}{mc}F^{\mu\nu}u_\nu \end{aligned} \tag{22}$$

(ii) To write this equation in terms of three-dimensional vectors we use the component forms

$$u^\mu = (\gamma c, \gamma\vec{v}) = (\frac{\mathcal{E}}{mc}, \frac{\vec{p}}{m}) \,, \tag{23}$$

and

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{24}$$

so that the $\mu = 0$ component of (22) becomes (using $\frac{dt}{d\tau} = \gamma$)

$$\begin{aligned} & \frac{1}{mc}\frac{d\mathcal{E}}{dt}\frac{dt}{d\tau} = -\frac{q}{mc}F^{0i}\frac{p_i}{m} \\ \Rightarrow \qquad & \frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{u} \,. \end{aligned} \tag{25}$$

The $\mu = i$ component of (22) gives

$$\frac{\gamma}{m}\frac{dp^i}{dt} = \frac{q}{mc}\left(F^{i0}u_0 + F^{ij}u_j\right) \tag{26}$$

$$\Rightarrow \qquad \frac{dp^i}{dt} = qE^i + \frac{q}{c}\epsilon^{ijk}B_k v_j \tag{27}$$

$$\Rightarrow \qquad \frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B} \,. \tag{28}$$

(iii) As there are no B-fields and $\vec{E} = (0, 0, E)$ we have in components

$$p_x = p_{0x} \,, \quad p_y = p_{0y} \,, \quad p_z = p_{0z} + qEt \,,$$

where p_{0x}, p_{0y} and p_{0z} are constants. Now recalling that $\vec{p} = \gamma m\vec{v}$ we have that at $t = 0$

$$p_{0x} = 0 \,, \quad p_{0y} = 0 \,, \quad p_{0z} = 0 \,.$$

Further recall that the kinetic energy of the particle (neglecting the field contribution) is

$$\begin{aligned}\mathcal{E}_{\rm kin} &= \sqrt{m^2c^4 + c^2|\vec{p}|^2} \\ &= c\sqrt{m^2c^2 + (qEt)^2}\,,\end{aligned}$$

so that

$$\vec{v} = \frac{\vec{p}}{\gamma m} = \frac{c^2\vec{p}}{\mathcal{E}_{\rm kin}}$$

which implies that

$$\frac{dz}{dt} = \frac{cqEt}{\sqrt{m^2c^2 + (qEt)^2}}$$

or upon integrating

$$z = \frac{c}{qE}\sqrt{m^2c^2 + (qEt)^2} + z_0$$

where $z_0 = -\frac{mc^2}{qE}$ is the constant of integration.

Problem 4

(i) We have that

$$\begin{aligned}\frac{\partial(F^{\rho\sigma}F_{\rho\sigma})}{\partial(\partial_\mu A_\nu)} &= 2F^{\rho\sigma}\frac{\partial(F_{\rho\sigma})}{\partial(\partial_\mu A_\nu)} \\ &= 2F^{\rho\sigma}\frac{\partial(\partial_\rho A_\sigma - \partial_\sigma A_\rho)}{\partial(\partial_\mu A_\nu)} \\ &= 2F^{\rho\sigma}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) \\ &= 4F^{\mu\nu} \qquad (29)\end{aligned}$$

as required. From the definition of the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ we have

$$\begin{aligned}\partial_\mu\tilde{F}^{\mu\nu} &= \partial_\mu(\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)) \\ &= \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma \\ &= 0 \qquad (30)\end{aligned}$$

due to the antisymmetry of the ϵ tensor.

(ii) The Proca Lagrangian is

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu} + \frac{m^2}{8\pi}A^\mu A_\mu\,. \qquad (31)$$

We can use the Euler-Lagrange equations,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0\,. \qquad (32)$$

to show that

$$\partial_\mu F^{\mu\nu} = -m^2A^\nu\,. \qquad (33)$$

(iii) The canonical stress energy tensor is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}_{\mathrm{P}}}{\partial(\partial_\mu A_\lambda)}(\partial^\nu A_\lambda) - g^{\mu\nu}\mathcal{L}_{\mathrm{P}} \\ &= -\frac{1}{4\pi}F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{16\pi}g^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} - \frac{1}{8\pi}g^{\mu\nu}m^2 A_\lambda A^\lambda \,. \end{aligned} \tag{34}$$

Now we consider

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -\frac{1}{4\pi}(\partial_\mu F^{\mu\lambda})\partial^\nu A_\lambda - \frac{1}{4\pi}F^{\mu\lambda}(\partial_\mu\partial^\nu A_\lambda) + \frac{1}{8\pi}(\partial^\nu F_{\lambda\rho})F^{\lambda\rho} - \frac{m^2}{4\pi}(\partial^\nu A_\lambda)A^\lambda \\ &= 0 \end{aligned} \tag{35}$$

where we use the definition of $F_{\mu\nu}$.

Bonus Problem

Starting from the sine-Gordon Lagrangian,

$$\mathcal{L}_{\mathrm{sG}} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + g^2(\cos\phi - 1) \tag{36}$$

we find:

1. The equation of motion is

$$\partial^\alpha\partial_\alpha\phi + g^2\sin\phi = 0\,. \tag{37}$$

2. With the ansatz,

$$\phi(x^0, x^1) = a\arctan\exp\Big[b\gamma(x^1 - \tfrac{v}{c}x^0)\Big]\,, \tag{38}$$

the equations of motion become

$$-b^2\phi'' + g^2\sin\phi = 0 \tag{39}$$

where now $\phi = a\tan^{-1}e^x$, or equivalently

$$\frac{b^2 a}{2}\mathrm{sech}\,x\tanh x + g^2\sin(a\tan^{-1}e^x) = 0\,. \tag{40}$$

One way to determine the constants is to expand in small x and match coefficients.

$$\begin{aligned} g^2\sin\tfrac{a\pi}{4} + \tfrac{1}{2}a\left(b^2 + g^2\cos\tfrac{a\pi}{4}\right)x - \tfrac{1}{8}(a^2g^2\sin\tfrac{a\pi}{4})x^2 \\ - \frac{a}{48}\left(20b^2 + (4+a^2)g^2\cos\tfrac{a\pi}{4}\right)x^3 + \mathcal{O}(x^4) = 0 \end{aligned} \tag{41}$$

Using the first few terms we find $b = \pm g$ and $a = \pm 4$. Substituting these values in it is straightforward to check the equations of motion hold for all values of x.

3. The Noether current is given by

$$J^\alpha = \Big[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial_\nu\phi - \delta^\alpha_\nu\mathcal{L}\Big]\frac{\partial X^\nu}{\partial\epsilon} - \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\frac{\partial\Phi}{\partial\epsilon}\,. \tag{42}$$

For the transformation given in the question this becomes

$$J^\alpha_\beta = \partial^\alpha\phi\partial_\beta\phi - \delta^\alpha_\beta\mathcal{L}\,. \tag{43}$$

Evaluating on the solution ansatz,

$$J^0_0 = 4g^2\gamma^2\mathrm{sech}^2(g\gamma(x^1 - \tfrac{v}{c}x^0)) = -(\tfrac{c}{v})J^1_0 = (\tfrac{c}{v})J^0_1 = -(\tfrac{c}{v})^2 J^1_1 \tag{44}$$

4. From the previous part

$$\mathcal{P}^0 = 4g^2\gamma^2 \mathrm{sech}^2 g\gamma x^1 \tag{45}$$

where as the quantity is conserved we can evaluate it at any time and we choose $x^0 = 0$. This quantity is the energy density of the solution. Integrating over all space we find

$$\int dx^1\, \mathcal{P}^0 = 8g\gamma \tag{46}$$

where we chose the positive root.