# Sample Exam

1. **(SI)** A spherical shell of radius $a$ is held at a fixed potential $V(\cos\theta)$. The general expression for the scalar potential, $\Phi(r,\theta,\phi)$, away from the shell is given by

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}[A_\ell r^\ell + B_\ell r^{-(\ell+1)}]P_\ell(\cos\theta) \ .$$

where $P_\ell(\cos\theta)$ are Legendre polynomials and $A_\ell$, $B_\ell$ are constants determined by the boundary conditions.

   (a) Why does the general expression not depend on the azimuthal angle $\phi$? What are physically sensible boundary conditions for the scalar potential at i) $r = 0$, ii) $r \to \infty$?

   (b) If the potential distribution on the spherical shell is given by

$$\Phi(a,\theta,\phi) = kE_0 a\cos\theta \ ,$$

   find the potential inside and outside the shell.

   (c) If the shell is split into two, with the upper hemisphere maintained at constant potential $+V$ and the lower at $-V$, find the coefficient of $P_\ell$ with $\ell = 0, 1, 2$ in the series solution for the potential inside the sphere.

Hint: You may use the orthogonality condition

$$\int_{-1}^{+1} P_m(\cos\theta)P_n(\cos\theta)\ d(\cos\theta) = \frac{2\delta_{n,m}}{2n+1} \ .$$

2. **(SI)** The Lagrangian density $\mathcal{L}(n_i,\ \partial_\mu n^i, \lambda)$ describing the 1+1 dimensional model,

$$\mathcal{L} \ = \ \tfrac{1}{2}\sum_{i=1}^{3}\ \partial_\mu n_i\ \partial^\mu n_i \ + \ \tfrac{1}{2}\lambda\,(\sum_{i=1}^{3} n_i n_i - 1) \ ,$$

is a function of the triplet of scalar fields $n_i(x)$, $i = 1, 2, 3$, their derivatives $\partial_\mu n_i = \partial n_i/\partial x^\mu$, and the constant $\lambda$.

   (a) By using the Euler-Lagrange equations for the scalar fields $n_i(x^\mu)$ show that the canonical stress tensor

$$T^{\mu\nu} \ = \ \sum_{i=1}^{3}\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu n_i\right)}\,\partial^\nu n_i \ - \ \eta^{\mu\nu}\mathcal{L}\,.$$

   satsifies $\partial_\mu T^{\mu\nu} = 0$ .

   (b) Consider the transformation

$$n_i' = \sum_{j=1}^{3} M_{ij} n_j$$

   where $M_{ij}$ are the entries of a constant orthogonal matrix. Show that

$$\mathcal{L}' \equiv \mathcal{L}(n_i') = \mathcal{L}(n_i)$$

   i.e. the transformation is a symmetry of the Lagrangian.

(c) Given the same Lagrangian as above but now with $\lambda \equiv \lambda(x)$ a scalar field show that the equations of motion for the $n_i(x)$ fields can be written as

$$\partial_\mu \partial^\mu n_i + \left( \sum_{i=1}^{3} \partial_\mu n_i \partial^\mu n_i \right) n_i = 0 \; .$$

3. **(G)** A Lagrangian describing a particle of mass $m$, charge $q$ and four velocity $u^\mu = \frac{dx^\mu}{d\tau}$ interacting with an electromagnetic four potential $A_\mu$, where $\mu \in \{0, 1, 2, 3\}$, can take the form

$$L \;=\; -\frac{m}{2} u^\mu u_\mu - \frac{q}{c} u^\mu A_\mu \, .$$

(a) Derive an equation of motion for the particle in terms of the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .

(b) Writing $F^{\mu\nu}$ in terms of fields $\vec{E}$ and $\vec{B}$

$$F^{\mu\nu} \;=\; \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \, ,$$

derive the three dimensional form of the particle's equations of motion.

(c) Consider the above particle in a static, uniform electric field $\vec{E} = (0, 0, E)$. Solve for the third, i.e. the $z$, component of the particle's position as a function of time, such that at $t = 0$ the velocity and position are given by

$$\vec{v}(t=0) = (0,0,0) \quad \text{and} \quad \vec{x}(t=0) = (0,0,0) \; .$$

You may find the integral

$$\int dx \; \frac{x}{\sqrt{A + x^2}} = \sqrt{A + x^2}$$

useful.

4. **(G)** Consider the covariant field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the four-potential $A_\mu$.

(a) Show that

$$\frac{\partial (F^{\rho\sigma} F_{\rho\sigma})}{\partial(\partial_\mu A_\nu)} = 4 F^{\mu\nu}$$

and that the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ satisfies

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \; .$$

(b) Show by using the Euler-Lagrange equations that the Proca Lagrangian involving a "mass" term for the gauge field

$$\mathcal{L}_{\rm P} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{8\pi} A_\mu A^\mu$$

with $m$ a constant parameter, produces the equations of motion

$$\partial_\mu F^{\mu\nu} = -m^2 A^\nu \; .$$

(c) For the Proca theory, calculate the canonical stress tensor,

$$T_{\mathrm{P}}^{\mu\nu} = \frac{\partial \mathcal{L}_{\mathrm{P}}}{\partial(\partial_\mu A_\lambda)}(\partial^\nu A_\lambda) - g^{\mu\nu}\mathcal{L}_{\mathrm{P}} \ .$$

and prove the differential conservation law

$$\partial_\mu T_{\mathrm{P}}^{\mu\nu} = 0 \ .$$

5. **Extra Question**
In this problem we will consider a scalar field theory in two-dimensional space-time with coordinates $x^\alpha$ where now $\alpha = 0, 1$. The space-time metric is

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

The Lagrangian for our theory is

$$\mathcal{L}_{\mathrm{sG}} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + g^2(\cos\phi - 1)$$

where $g$ is a constant parameter. This the well known sine-Gordon theory.

(a) Calculate the equation of motion for $\phi$.

(b) Consider the ansatz

$$\phi(x^0, x^1) = a \arctan \exp\Big[b\gamma(x^1 - \frac{v}{c}x^0)\Big] \ ,$$

where $v$, corresponding the velocity of the solutions, is a specified constant. For what values of $a$ and $b$ is this a solution. (Here, as usual, $\gamma^{-2} = 1 - \frac{v^2}{c^2}$).

(c) The Lagrangian is invariant under the transformations

$$\bar{x}^\alpha \to x^\alpha = \bar{x}^\alpha + \epsilon\delta^\alpha_\beta \ , \quad \bar{\phi}(\bar{x}) \to \phi(x) = \bar{\phi}(\bar{x})$$

for constant $\epsilon$ and $\beta = 0, 1$. Calculate the corresponding Noether currents and evaluate them on the above solution.

(d) For the case $\beta = 0$ let us call the Noether current $\mathcal{P}^\alpha$. Calculate

$$\int_{-\infty}^{\infty} dx^1 \ \mathcal{P}^0 \ ,$$

for the above solution.