

Vectors III

MA1S1

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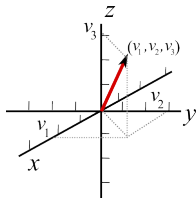
October 17, 2014

Anton & Rorres: Ch 3.3

Hefferon: Ch One, sec II.1 and II.2

Vector in Space

Every vector \mathbf{v} can be drawn as an arrow starting from the origin $(0, 0, 0)$ and it is then uniquely determined by the coordinates of its terminal point.



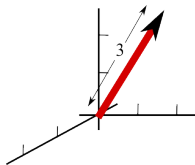
We can thus write three-dimensional vectors as 3-tuples: $\mathbf{v} = (v_1, v_2, v_3)$. Alternatively we can write them in terms of **orthonormal** (orthogonal and normalised to have unit length) basis vectors: $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

The length of a vector is given by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Question

Consider vectors in \mathbb{R}^3 with norm 3 in the canonical position i.e. with their end points at the origin.



What is the geometrical set of end points of such vectors?

- A. A plane.
- B. A line.
- C. A sphere.

Dot product

The **dot product** of two 3-dimensional vectors \mathbf{v} and \mathbf{w} is given in terms of the components $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ to be

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$$

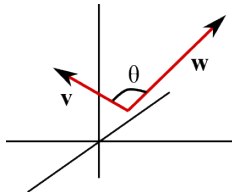
i.e. the number you get by multiplying the first component of \mathbf{v} by the first component of \mathbf{w} , the second by the second, the third by the third **and then adding** these numbers together.

So if (say) $\mathbf{v} = 11\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$, we get

$$\mathbf{v} \cdot \mathbf{w} = 11(3) + (-2)(4) + 5(0) = 25$$

The geometric interpretation is the same as in two-dimensions

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$



Rules for scalar product

This scalar product satisfies all the same properties as the two-dimensional case. For example

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$$

Check the others! Also the Cauchy-Schwarz inequality holds and so the Triangle inequality.

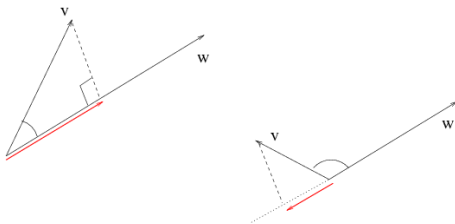
We can also easily see that as in the plane, the scalar product of a vector with itself is the norm of the vector squared:

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

As the norm of a vector has a geometric interpretation as the length between its start and finish points hence we can derive a formula for the distance between two points in space.

Projection of a vector along another vector

Consider vectors \mathbf{v} and \mathbf{w} . Dropping a perpendicular line from the end of \mathbf{v} to the line along \mathbf{w} gives the end point of the projection.



The “length” of the projection will be $\|\mathbf{v}\| \times |\cos \theta|$ where θ is the angle between the two vectors.

Projection of a vector along another vector

To find the projection vector, we need to multiply \mathbf{w} by the right factor to make its length $\|\mathbf{v}\| \cos \theta$. Start by making a unit vector with the same direction as \mathbf{w} by dividing by its length. So

$$\hat{\mathbf{w}} = \frac{\mathbf{w}}{\|\mathbf{w}\|}$$

is the unit vector in the direction of \mathbf{w} . Then the projection is

$$\|\mathbf{v}\| \cos \theta \hat{\mathbf{w}} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|} \left(\frac{\mathbf{w}}{\|\mathbf{w}\|} \right)$$

This formula is also correct in the case of obtuse angles θ , when $\cos \theta < 0$, when the projection will be in the opposite direction to \mathbf{w} .

Tidying up the above, we get that the projection along \mathbf{w} of \mathbf{v} is the vector

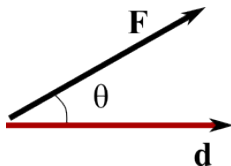
$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w}$$

An examples of the scalar product in physics

We will take an example from physics:

As we have described forces and displacement are both described by vectors in space.

What is the work done by a constant force \mathbf{F} on an object that undergoes a displacement \mathbf{d} ?



The answer

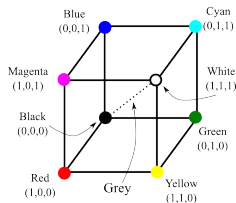
$$W = \mathbf{F} \cdot \mathbf{d} = Fd \cos \theta .$$

So if a particle moves perpendicular to an external force no work is done on it by that force.

An example of the scalar product in computer graphics

Colours in computer graphics are commonly based on the RGB colour model whereby arbitrary colours are created by adding together percentages of the primary colours **R**ed, **G**reen and **B**lue.

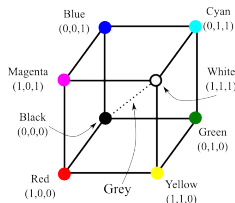
We can identify each of the primary colours with one of the basis vectors



and then every colour, \mathbf{c} , corresponds to a vector in this colour space

$$\mathbf{c} = (c_1, c_2, c_3) .$$

An example of the scalar product in computer graphics



By taking the scalar product of our colour vector \mathbf{c} with the reference vector

$$\mathbf{g} = \frac{1}{3}(1, 1, 1)$$

we find the component of the colour vector lying along the “grey diagonal” and so the grey-scale intensity can be defined

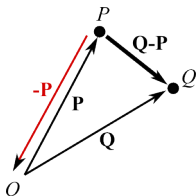
$$I = \mathbf{c} \cdot \mathbf{g} .$$

In fact the human eye is more sensitive at frequencies corresponding to green light and less to red and blue so the reference vector is sometimes taken to be

$$\mathbf{g}' = (0.3, 0.59, 0.11) .$$

Distance formula in space

If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are two points in space, we consider their position vectors (represented by the arrows from the origin to P , \overrightarrow{OP} and to Q , \overrightarrow{OQ}). We saw before that $\mathbf{Q} - \mathbf{P}$ is the vector represented by the arrow from P to Q , \overrightarrow{PQ} .



But we can see then that the length $\|\mathbf{Q} - \mathbf{P}\|$ of the vector $\mathbf{Q} - \mathbf{P}$ must be exactly the distance from P to Q . So we get

$$\begin{aligned}\text{dist}(P, Q) &= \|\mathbf{Q} - \mathbf{P}\| \\ &= \|(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\| \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

An Example

Cross products

We previously described the the scalar product which takes two vectors and forms a number or a scalar:

$$(\mathbf{v}, \mathbf{w}) \in V \times V \mapsto \mathbf{v} \cdot \mathbf{w} \in \mathbb{R} .$$

In three dimensions there is notion of a product of two vectors which gives back a vector.

Definition: The **cross product** $\mathbf{v} \times \mathbf{w}$ of two vectors

$$\begin{aligned}\mathbf{v} &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}, \\ \mathbf{w} &= w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k},\end{aligned}$$

is

$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \mathbf{i} + (v_3 w_1 - v_1 w_3) \mathbf{j} + (v_1 w_2 - v_2 w_1) \mathbf{k}$

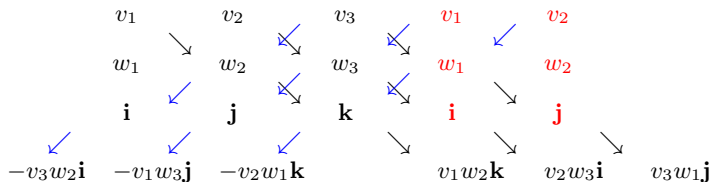
There is a sort of ‘easy’ way to remember the formula. Write a table

v_1	v_2	v_3
w_1	w_2	w_3
i	j	k

and extend it by repeating the first two columns to the right, like this

v_1	v_2	v_3	v_1	v_2
w_1	w_2	w_3	w_1	w_2
i	j	k	i	j

Then take ‘forward diagonal’ products with plus signs and backwards diagonals with minus in front



and add these vectors.

There is actually a pattern here: The first component of $\mathbf{v} \times \mathbf{w}$

$$(v_2w_3 - v_3w_2) = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}$$

depends on the components of \mathbf{v} and \mathbf{w} other than the first. Starting with this first component we can get to the next component by adding 1 to the subscripts and interpreting $3 + 1$ as 1 i.e. thinking of cycling the subscripts around $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ to get the next component:

$$v_2w_3 - v_3w_2 \mapsto v_3w_1 - v_1w_3$$

Repeating again we get the third component.

So you only have to remember the first component and the others are automatic.

Properties of cross products

- (i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.

Properties of cross products

- (i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.
- (ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$

Proof.

Recall that:

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}$$

from this we can see the **antisymmetry** property.



Properties of cross products

- (i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.
- (ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
- (iii) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both \mathbf{v} and \mathbf{w} .

Proof.

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) &= (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \cdot ((v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}) \\&= v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) \\&= v_1v_2w_3 - v_1v_3w_2 + v_2v_3w_1 - v_2v_1w_3 + v_3v_1w_2 - v_3v_2w_1 \\&= 0\end{aligned}$$

because if you look carefully everything cancels with something else.

So $\mathbf{v} \perp \mathbf{v} \times \mathbf{w}$.

To show $\mathbf{w} \perp \mathbf{v} \times \mathbf{w}$, we can either repeat a similar calculation or we can use $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \perp \mathbf{w}$.



Properties of cross products

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- (ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
- (iii) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both \mathbf{v} and \mathbf{w} .
- (iv) $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2$.

Proof.

The proof starts with the definition of the norm of cross-product:

$$\|\mathbf{v} \times \mathbf{w}\|^2 = (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2$$

and that of the rhs

$$\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) - (v_1 w_1 + v_2 w_2 + v_3 w_3)^2$$

multiply out both sides and compare. □

It is interesting to note that as $(\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \cos^2 \theta$ where θ is the angle between the \mathbf{v} and \mathbf{w} by using the identity $1 - \cos^2 \theta = \sin^2 \theta$ we find that

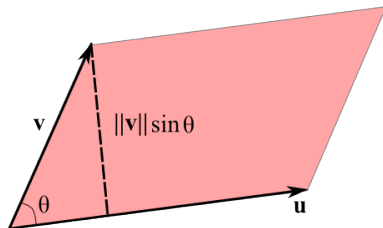
$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 \\ &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \sin^2 \theta . \end{aligned}$$

Furthermore as $0 \leq \theta \leq \pi$ we have that $\sin \theta \geq 0$ so that we can take the square-root of both sides

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta .$$

Area of a Parallelogram

This gives us a geometric interpretation of the magnitude of the cross-product of two vectors.



As the area of a parallelogram is given by

$$A = (\text{base})(\text{altitude}) = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

we see that the norm of the cross product gives the area of the parallelogram formed by the two vectors. Also the resultant vector is normal to the surface of the parallelogram (see in a bit).

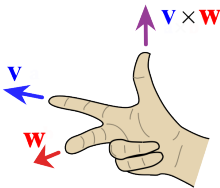
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- (iv) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$ where θ is the angle between \mathbf{v} and \mathbf{w} .
- (v) The vector $\mathbf{v} \times \mathbf{w}$ points in a direction that follows from using the right-hand rule.

If the angle θ between \mathbf{v} and \mathbf{w} is not 0 and not π , then the vectors \mathbf{v} and \mathbf{w} are not in the same direction and also not in exactly opposite directions. So as long as $0 < \theta < \pi$, then we can say that there is one plane through the origin parallel to both \mathbf{v} and \mathbf{w} (or containing both vectors if we draw them from the origin). The cross product is then in one of the two normal directions to that plane.

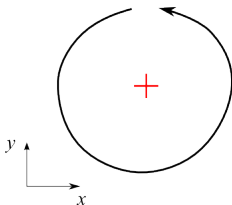
In the case $0 < \theta < \pi$, we can describe the cross product up to one of the two normal directions to the plane.

This can be described in terms of the directions of the index finger, middle finger and thumb on your right hand if you hold them perpendicular to one another.



Another way is to place a corkscrew (an ordinary right-handed corkscrew) along the vertical axis and twist from the x -axis towards the y -axis. It should travel in the direction of the positive z -axis.

counter-clockwise = positive



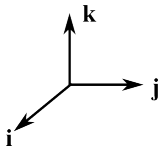
Also known as

“Righty tighty, lefty loosey”

Imaging a corkscrew placed so it is perpendicular to the plane of \mathbf{v} and \mathbf{w} . Turn the screw from \mathbf{v} towards \mathbf{w} and the direction it travels is the same as the direction of $\mathbf{v} \times \mathbf{w}$.

Basis vectors

It is worth noticing that the standard basis, \mathbf{i} , \mathbf{j} , \mathbf{k} , is right-handed.

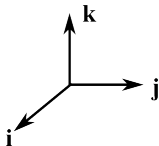


What is:

$$\mathbf{i} \times \mathbf{j} = ?$$

Basis vectors

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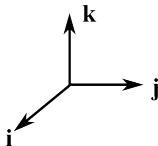


What is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

Basis vectors

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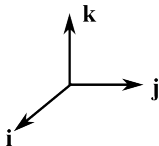
What is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = ?$$

Basis vectors

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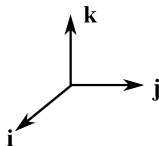
What is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

Basis vectors

It is worth noticing that the standard basis, \mathbf{i} , \mathbf{j} , \mathbf{k} , is right-handed.



What is:

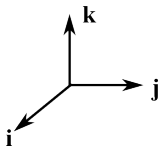
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = ?$$

Basis vectors

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What is:

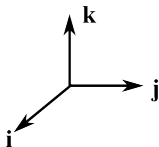
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Basis vectors

It is worth noticing that the standard basis, \mathbf{i} , \mathbf{j} , \mathbf{k} , is right-handed.



What is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

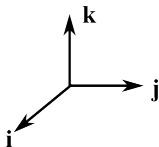
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Finally $\mathbf{i} \times \mathbf{i} = ?$

Basis vectors

It is worth noticing that the standard basis, \mathbf{i} , \mathbf{j} , \mathbf{k} , is right-handed.



What is:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

Finally $\mathbf{i} \times \mathbf{i} = 0 = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k}$.

Properties of cross products

- (i) $\mathbf{v} \times \mathbf{w}$ is a vector in space.
- (ii) $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
- (iii) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both \mathbf{v} and \mathbf{w} .
- (iv) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$ where θ is the angle between \mathbf{v} and \mathbf{w} .
- (v) The vector $\mathbf{v} \times \mathbf{w}$ points in a direction that follows from using the right-hand rule.
- (vi) Some algebraic properties of the cross product that are as you would expect for products:

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \\ (\mathbf{u} + \mathbf{v}) \times \mathbf{w} &= \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \\ (k\mathbf{v}) \times \mathbf{w} &= k(\mathbf{v} \times \mathbf{w}) \\ &= \mathbf{v} \times (k\mathbf{w})\end{aligned}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and any scalar $k \in \mathbb{R}$. (But recall that the order **does** matter: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.)

Vector triple products

Let us consider the cross-product of three vectors. Now the order in which we carry out the two products is important. That is

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

make sense, but that is usually not the same as $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

For example

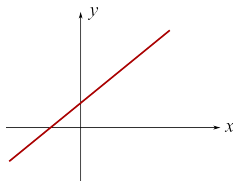
$$\begin{aligned}\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) &= \mathbf{i} \times \mathbf{k} \\ &= -\mathbf{j} \\ (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} &= \mathbf{0} \times \mathbf{j} \\ &= \mathbf{0}\end{aligned}$$

Equations of lines

Recall the general equation of a line in two-dimensions \mathbb{R}^2 has the form

$$Ax + By = C$$

where A and B can't be both zero.



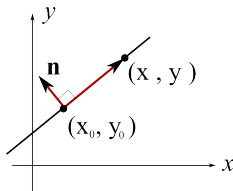
The y intercept is at $x = 0$ and is C/B while the x intercept is at $y = 0$ and is C/A . A line through the origin is given by the equation

$$Ax + By = 0 .$$

Essentially a line is specified by its slope and one point on the line.

Equations of lines

An alternative way to specify the slope of a line is to specify a vector which is orthogonal to the vector connecting any two points on the line.



This vector is called the **normal** vector. In fact given a reference point on the line P_0 with coordinates (x_0, y_0) the complete line can be described as the set of all points P with coordinates (x, y) such that the vector $\overrightarrow{P_0P}$ satisfies

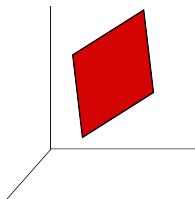
$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0 .$$

Or if give the components of the normal vector: $\mathbf{n} = (n_1, n_2)$ the set of solutions to the equation

$$n_1(x - x_0) + n_2(y - y_0) = 0 .$$

Equations of planes

We consider first the question of how we might describe the orientation of a plane in space.

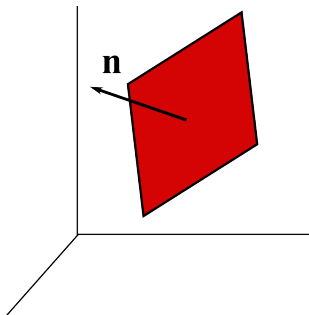


As for lines we could perhaps attempt to think about the slope of a plane as the slope of the steepest line up the plane, but this will not tell us the orientation of our plane in space. We can move the plane around (say rotating it around a vertical direction), keeping the same largest slope, but changing the way the plane is oriented in space.

Equations of planes

A better method is the analogue of second method used for the line above i.e picking a **normal** vector perpendicular to the plane.

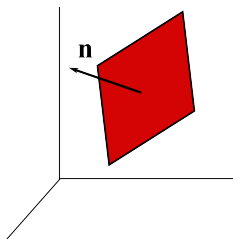
Consider a plane with a normal vector



If we move the plane around parallel to itself the normal points constant direction (or to be more precise, you could also turn the handle around the other way, so that it will reverse direction completely, and still have the board in the same plane.)

Equations of planes

Picking a vector perpendicular to the plane allows you to say which way a it is tilted in space.



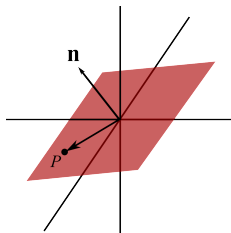
The normal vector is not unique — multiplying the vector by a positive or negative (but not zero) scalar will not change the orientation of a perpendicular plane.

Now if we think about all the planes perpendicular to a given (or fixed) normal vector, then we can see that we could pick out one of these many planes by knowing one point on the plane.

Equations of planes (through the origin)

Say \mathbf{n} is a normal vector to our plane. How do we come up with an equation that is satisfied by all the points on the plane?

The answer to this is a little easier to understand if we think about the case where the origin $(0, 0, 0)$ is one point on the plane. Say $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ is the normal vector and we think of any point $P = (x, y, z)$ on the plane.



In terms of position vectors, you should be able to see fairly easily that the position vector $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ should lie in the plane. So it should be perpendicular to \mathbf{n} . But a way to express that by an equation is

$$\mathbf{n} \cdot \mathbf{P} = 0.$$

Equations of planes (through the origin)

Writing that out $\mathbf{n} \cdot \mathbf{P} = 0$ using components we get

$$n_1x + n_2y + n_3z = 0$$

for the equation of a plane *through the origin* $(0,0,0)$. You can see that $x = 0, y = 0, z = 0$ is one solution of this equation.

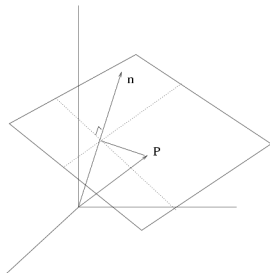
We can think of this in reverse given any **linear equation** (actually a *homogeneous linear equation* because of the constant term being 0)

$$ax + by + cz = 0$$

i.e. no terms with x^2, xy, z^3 , etc. this is the equation for a plane through the origin with normal $\mathbf{n} = (a, b, c)$.

Equations of planes

To deal with a plane that does not go through the origin, we can see what to do by using projections. Take any plane perpendicular to \mathbf{n} , a normal vector drawn from the origin through the plane,



If you look geometrically, you will see that the vector projection

$$\text{proj}_{\mathbf{n}}(\mathbf{P})$$

must be the same no matter what point P in the plane you try. That means

$$\frac{\mathbf{P} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n}$$

is always the same vector for all points P on the plane.

Equations of planes

This translates into a single equation

$$\mathbf{P} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{P} = d, \text{ where } d \text{ is a constant.}$$

If we write that out in components of $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ we get the equation

$$n_1x + n_2y + n_3z = d.$$

This equation means that for a given n_1, n_2, n_3 and d that every point (x, y, z) that satisfies the above equation lies in the plane.

Alternatively given an equation

$$ax + by + cz = d$$

this describes a plane where $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a (fixed) nonzero vector perpendicular to the plane and d is a constant related to the intercept point of the plane with the coordinate axes.

An example

Another example

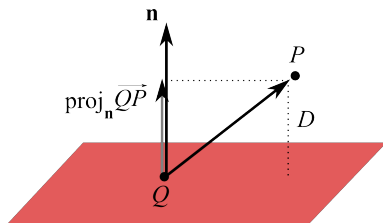
Another example continued

Note also that we were ‘lucky’ here that the points were on the axes and the 3 simultaneous equations we got were easy to figure out.

We’ll learn later an efficient method to solve linear simultaneous equations with possibly large numbers of unknowns. When we know that we’ll be able to do more complicated points. As long as 3 points in space are not collinear, there will be just one plane that goes through them.

Geometrical Application

How do we find the distance between a point P with coordinate (x_0, y_0, z_0) and a plane with the equation $ax + by + cz = d$?



Let $Q(x_1, y_1, z_1)$ be any point on the plane. Move the normal vector, \mathbf{n} , such that its initial point is at point Q . The distance to the point P is the length of the projection of $\overrightarrow{QP} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ onto the normal vector $\mathbf{n} = (a, b, c)$.

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{QP}\| = \frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Geometrical Application

We know that $\|\mathbf{n}\| = \sqrt{a^2 + b^2 + c^2}$ and

$$\overrightarrow{QP} \cdot \mathbf{n} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$

so that

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

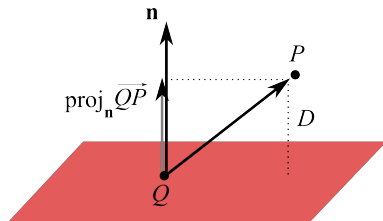
As the point Q lies in the plane we have the relation:

$$ax_1 + by_1 + cz_1 = d$$

so that ...

Geometrical Application

How do we find the distance between a point P with coordinate (x_0, y_0, z_0) and a plane with the equation $ax + by + cz = d$?

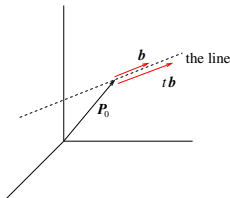


Answer :

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Lines in space

Returning to the case of lines we note that we can describe a line in space, \mathbb{R}^3 , by giving a (nonzero) vector \mathbf{b} parallel to the line which describes its orientation and one reference point $P_0 = (x_0, y_0, z_0)$ on the line.



One can quite easily see that the points with position vectors

$$\mathbf{x} = \mathbf{P}_0 + t\mathbf{b}$$

all lie on the line through the point \mathbf{P}_0 in the direction parallel to \mathbf{b} . As t varies through all of \mathbb{R} (positive, zero and negative values) we get all the points on that line and we get each point only once.

We refer to t as a *parameter* and the above equation $\mathbf{x} = \mathbf{P}_0 + t\mathbf{b}$ as a parametric equation for the line in vector form.

If $\mathbf{P}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is the position vector of the point $P_0 = (x_0, y_0, z_0)$ on the line, if $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and if we write $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then what we have is in components

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + t(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

or we can write this as

$$(x, y, z) = (x_0 + tb_1, y_0 + tb_2, z_0 + tb_3)$$

which is common form for plotting in computer packages.

Lines in Space

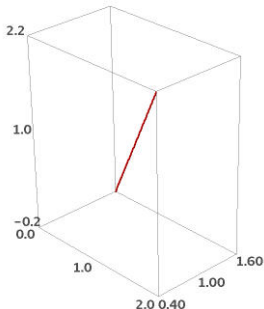
Here we want to plot a line parametrically. We pick a starting point $(1, 1, 1)$ and a vector $(0.5, -0.3, 1.3)$ and a parameter t . In SAGE we must specify that t is a *variable*.

```
t=var('t')
```

The SAGE command is `parametric_plot`. We will let the parameter run from -2 to 2 and imagine the rest. We will also choose the line to be red and make it thick enough to see.

```
parametric_plot((1+0.5*t, 1-0.3*t, 1+1.3*t),  
(t,-2,2),color="red",thickness=3)
```

[Sleeping...](#) [Make Interactive](#)



Parametric equations of a line

The vector relation

$$(x, y, z) = (x_0 + tb_1, y_0 + tb_2, z_0 + tb_3)$$

can be thought of as 3 scalar equations

$$\begin{cases} x &= x_0 + b_1t \\ y &= y_0 + b_2t \\ z &= z_0 + b_3t \end{cases}$$

We call these the **parametric equations** for the line.

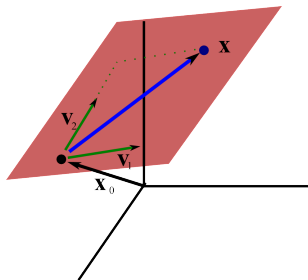
Notice that the right hand side is linear in t , the constant terms come from the coordinates (x_0, y_0, z_0) of a point on the line, and the coefficients of t give the components of a vector $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ parallel to the line.

Example

Another example

Parametric Planes

Previously we gave a **point-normal** equation for a plane i.e. by specifying a point on the plane and a normal vector. Alternatively we can specify a point on the plane with position vector \mathbf{x}_0 and two parallel, non-collinear vectors \mathbf{v}_1 and \mathbf{v}_2 .



Then every other point on the plane with position vector \mathbf{x} can be written as

$$\mathbf{x} = \mathbf{x}_0 + t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 ,$$

in terms of the **two** parameters t_1 and t_2 .

Parametric Planes: Example

Cartesian equations of lines

Starting with parametric equations for a line

$$\begin{cases} x &= x_0 + b_1 t \\ y &= y_0 + b_2 t \\ z &= z_0 + b_3 t \end{cases}$$

we can solve each of the three equations for the parameter t in terms of all the rest of the quantities. We get

$$t = \frac{x - x_0}{b_1}, \quad t = \frac{y - y_0}{b_2}, \quad t = \frac{z - z_0}{b_3}.$$

(At least we can do this if none of b_1 , b_2 or b_3 is zero.) If (x, y, z) is a point on the line, there must be one value of t that gives the point (x, y, z) from the parametric equations, and so the 3 values for t must coincide. We get

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}.$$

These equations are called *cartesian equations for the line*.

You might wonder what sort of equation we have with two equalities in it? Well it is two equations

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2} \text{ and } \frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}.$$

Each of these two equations is a linear equation, and so the equation of a plane. So we are representing the line as the intersection of two particular planes when we write the cartesian equations.

The first plane

$$\frac{x - x_0}{b_1} = \frac{y - y_0}{b_2}$$

could be rewritten

$$\frac{1}{b_1}x - \frac{1}{b_2}y + 0z = \frac{x_0}{b_1} - \frac{y_0}{b_2}$$

so that it looks like $ax + by + cz = d$.

We see that there is no z term in the equation, or the normal vector $(1/b_1, -1/b_2, 0)$ is horizontal. This means that the plane is parallel to the z -axis and is the vertical plane that contains the line with which we started.

We could similarly figure out that the second plane

$$\frac{y - y_0}{b_2} = \frac{z - z_0}{b_3}$$

is the plane parallel to the x -axis that contains our line.

Now, there is an aspect of this that is a bit untidy. From the cartesian equations we can see that

$$\frac{x - x_0}{b_1} = \frac{z - z_0}{b_3}$$

follows immediately, and this is the equation of the plane parallel to the y -axis that contains our line.

It is arbitrary that we pick out the two planes parallel to the z -axis and the x -axis as those that contain our line, while discriminating against the y -axis.

We could take the two planes to be any two of the planes containing the line and parallel to the x , y and z -axes.

Example

Another Example

Another Example

Scalar triple product

We can combine the two products for vectors, the dot product and the cross product:

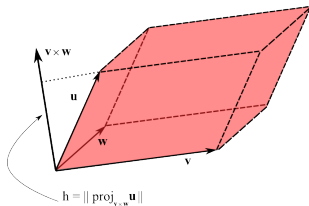
Given three vectors \mathbf{u} , \mathbf{v} and \mathbf{w} we can form

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$$

This has a different interpretation as the determinant of a 3×3 matrix

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Geometrically we can think of the scalar triple product as the volume of the parallelepiped formed by the three vectors \mathbf{u} , \mathbf{v} and \mathbf{w} .



The area of the base is given from before by the norm of the cross product $\mathbf{v} \times \mathbf{w}$. The height of parallelepiped is the length of the projection of \mathbf{u} onto to the vector perpendicular to the base i.e. $\mathbf{v} \times \mathbf{w}$. This is

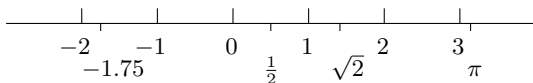
$$\begin{aligned} h &= \|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\| \\ &= \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|} . \end{aligned}$$

Now the volume is $h \times (\text{area of base})$ or

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| .$$

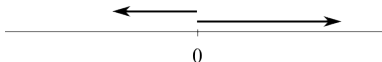
Higher dimensions

We have used the notation \mathbb{R} as a special symbol to signify the set of all real numbers. We are used to picturing \mathbb{R} as the set of all points on an axis or number line.

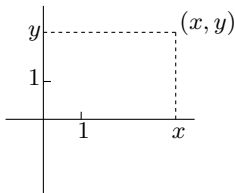


You should realise that the idea is that the line is all filled up by the real numbers (no gaps). Each real number has its point on the line to represent it and each point corresponds to some number. This is one dimension. We know how to add and multiply numbers.

We can think of this as a one dimensional vector space.



By \mathbb{R}^2 we mean the set of all ordered pairs (x, y) of real numbers and we have a graphical (or pictorial) way of looking at this as the set of all points in a plane.



We can alternatively think of vectors $x\mathbf{i} + y\mathbf{j}$ in two dimensions (as a picture of \mathbb{R}^2). Using mathematical set notation

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

(which reads as “ \mathbb{R} two equals the set of all ordered pairs (x, y) such that x and y are elements of the set \mathbb{R} , the set of real numbers”).

By \mathbb{R}^3 we mean the set of ordered triples (x, y, z) or real numbers. That is

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

and we are now used to the idea that these triples can be pictured by either looking at points (x, y, z) in space, or looking at vectors (arrows) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space.

We can think in terms of coordinates of points, or of the components of vectors, and we have on several occasions realised that it can be convenient to switch from a point to its position vector. When we manipulate vectors, we can think graphically about what we are doing, but it is usually much easier to calculate via components.

When you look at the progression from \mathbb{R} to \mathbb{R}^2 to \mathbb{R}^3 formally, from real numbers to ordered pairs of real numbers to ordered triples, there does not seem to be anything to stop us going on to ordered 4-tuples (x_1, x_2, x_3, x_4) of real numbers, or to ordered 5-tuples $(x_1, x_2, x_3, x_4, x_5)$.

Mathematically we can just define

$$\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$$

and

$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\}$$

In fact, for any $n = 2, 3, 4, \dots$ we let \mathbb{R}^n be the set of all n -tuples of real numbers. By an n -tuple (x_1, x_2, \dots, x_n) we mean a list of n numbers where the order matters. So

$$(1, 2, 3, 4, 5) \in \mathbb{R}^5$$

is different from

$$(2, 1, 3, 4, 5) \in \mathbb{R}^5.$$

In general then, two n -tuples

$$(x_1, x_2, \dots, x_n)$$

and

$$(y_1, y_2, \dots, y_n)$$

are to be considered equal only when they are absolutely identical. That means only when

$$x_1 = y_1, \text{ and } x_2 = y_2, \text{ etc, up to } \dots, \text{ and } x_n = y_n.$$

Points in different dimensions should not be compared. So $(1, 2) \in \mathbb{R}^2$ and $(1, 2, 0) \in \mathbb{R}^3$ are not the same.

Question: Why do we need \mathbb{R}^4 , \mathbb{R}^5 , and so on?

There is no good way to picture them. We can't really draw or even think in 4 dimensions, not in any satisfactory way. And 5 dimensions is even worse from that point of view. Nevertheless, there are many practical reasons why \mathbb{R}^n is useful even when n is very large.

One example is in computer graphics as we previously discussed. Each pixel has an x and y position, but also a RGB coordinate parameterised by three coordinates (sometimes *hue*, *saturation*, and *brightness*). Thus each pixel is a point in \mathbb{R}^5 : (x, y, h, s, b) .

As a different example suppose you are working for the consumer protection agency and you want to make sure that the milk sold in the shops is in accordance to what it claims to be on the label. What you might do is gather up a number of cartons of milk (say we stick to one size and brand) and do various measurements on each of those you collected. So one measures:

- volume of milk in the carton
- fat content
- calcium content
- temperature of the milk in the shop
- vitamin D

So, you would end up with 5 numbers for each carton of milk. It is not a big step to consider these numbers as giving one point in \mathbb{R}^5 for each sample. The kinds of techniques you would use to analyse the data are most naturally described in terms of manipulating points in \mathbb{R}^5 .

Indeed experimental data will often be described a vector in some high dimensional space.

We are now going to describe some of the basic kinds of manipulation we can do on points in \mathbb{R}^n (including $n = 5$). They will be exact parallels to what we did for vectors in \mathbb{R}^2 and \mathbb{R}^3 .

It may seem strange to treat points like vectors, but remember we already got into the habit of changing from points to their position vectors. And we do most calculations with the numbers, the components of the vectors. Finally, we can't see what we are doing in higher dimensions and so there is no reason to distinguish between points and vectors.

Arithmetic in \mathbb{R}^n

In \mathbb{R}^n we define the *sum* of two n -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

to be

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

If $k \in \mathbb{R}$ is a scalar and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then we define $k\mathbf{x}$ by the rule

$$k\mathbf{x} = (kx_1, kx_2, \dots, kx_n).$$

Example: If $\mathbf{x} = (1, 2, 3, 4)$ and $\mathbf{y} = (7, 8, 9, 10)$ are in \mathbb{R}^4 , then

$$\begin{aligned}5\mathbf{x} &= (5, 10, 15, 20) \\2\mathbf{y} &= (14, 16, 18, 20) \\5\mathbf{x} + 2\mathbf{y} &= (5 + 14, 10 + 16, 15 + 18, 20 + 20) \\&= (19, 26, 33, 40)\end{aligned}$$

We can see that there is no real problem following the rules to calculate these operations.

It is possible to check out that the operations on \mathbb{R}^n that we have defined obey the ‘standard’ rules (or axioms) of algebra. Example:

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n)) + (w_1, w_2, \dots, w_n) \\&= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) + (w_1, w_2, \dots, w_n) \\&= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \dots, (u_n + v_n) + w_n) \\&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \dots, u_n + (v_n + w_n)) \\&= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \dots, u_n + (v_n + w_n)) \\&= \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

Dot products in \mathbb{R}^n

We *define* the dot product of two elements (vectors or points as you like)

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

to be the number

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

So this is the same rule as we know from \mathbb{R}^2 and \mathbb{R}^3 extended to take in all the coordinates.

This scalar or dot product satisfies all the properties we discussed previously. E.g.

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} .$$

Which can easily be seen by using the definition. Furthermore if we define the zero vector

$$\mathbf{0} = (0, 0, \dots, 0)$$

then we have the properties $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$.

Norms and distances

We define the **norm** or **magnitude** of $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ as

$$\begin{aligned}\|\mathbf{x}\| &= \sqrt{\mathbf{x} \cdot \mathbf{x}} \\ &= \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\end{aligned}$$

This also allows us to define the notion of a distance in n -dimensional space. We define the **distance** between two points in n -dimensional space $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$\begin{aligned}\text{distance}(\mathbf{x}, \mathbf{y}) &= \|\mathbf{y} - \mathbf{x}\| \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2}\end{aligned}$$

Basis vectors

In \mathbb{R}^3 we defined the basis vectors

$$\mathbf{i} = (1, 0, 0) , \quad \mathbf{j} = (0, 1, 0) , \quad \mathbf{k} = (0, 0, 1) .$$

We can generalise this to n -dimensions

$$\mathbf{e}_1 = (1, 0, \dots, 0) , \quad \mathbf{e}_2 = (0, 1, \dots, 0) , \dots , \quad \mathbf{e}_n = (0, 0, \dots, 1) .$$

Now any vector in n -dimensions can be written as

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \dots + v_n \mathbf{e}_n .$$

Note that $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$ but $\mathbf{e}_1 \cdot \mathbf{e}_j = 0$ for $j = 2$ or 3 or \dots

We've seen that there is a geometric meaning behind the scalar product which is one of the reasons it plays such an important role in the physical world. But what about the scalar product in higher dimensions, a key use is in defining the **projection** of one vector onto another.

Here is a second example:

The **Universal Product Code** is the code associated to the bar code scanned at many shop's checkout counter. The code is a vector of length 12. In this case all the entries are integer so it corresponds to a vector \mathbf{u} in \mathbb{Z}^{12} . The first 11 digits correspond to manufacturer and product information, but the last digit is a check digit chosen so that $\mathbf{c} \cdot \mathbf{u} = 0$ where

$$\mathbf{c} = (3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1) .$$

The idea is that if there is an error in the code, a transposition or alteration, the vector will likely no longer be orthogonal to \mathbf{c} and so the error will be caught.

Theorem (Cauchy-Schwarz inequality)

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$-\|\mathbf{x}\|\|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

always holds. We can state this as

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$$

Proof.

Let us square the left hand side use the component notation:

$$\begin{aligned}(\mathbf{x} \cdot \mathbf{y})^2 &= (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 = x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + \cdots + x_n^2 y_n^2 \\&= x_1^2 y_1^2 + x_1^2 y_2^2 - x_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 + x_1^2 y_3^2 - x_1^2 y_3^2 \cdots + x_n^2 y_n^2 \\&= (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \\&\quad - x_1^2 y_2^2 + 2x_1 x_2 y_1 y_2 - x_2^2 y_1^2 + \dots \\&= \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (x_1 y_2 - x_2 y_1)^2 + \dots\end{aligned}$$

where in the last line every term in \dots is the negative of a square of a similar form e.g. $(x_1 y_3 - x_3 y_1)^2$ or $(x_2 y_5 - x_5 y_2)^2$. As all these squares are greater than or equal to zero we have that

$$|\mathbf{x} \cdot \mathbf{y}|^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$$



Note that in \mathbb{R}^2 and \mathbb{R}^3 we showed that

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta$$

where θ is the angle between the vectors \mathbf{x} and \mathbf{y} . We can't use that in \mathbb{R}^n when $n \geq 4$ because we can't see anything in higher dimensions. So we need proofs of theorems like the above that don't rely on imagining triangles in higher dimensional space.

Now that we have the inequality, we can make a definition of an angle.

Definition

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define the angle θ between \mathbf{x} and \mathbf{y} to be that $\theta \in [0, \pi]$ where

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

There is a snag in this definition. It really says as there is a potential problem of division by 0 when one of the points \mathbf{x} or \mathbf{y} is the origin and so has all coordinates 0 i.e. is the point

$$\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$$

As before this is just to say we must treat the zero vector as a special case

$$\mathbf{x} \cdot \mathbf{0} = 0 .$$

Example.

Definition

We call $\mathbf{x} \in \mathbb{R}^n$ *perpendicular* (or *orthogonal*) to $\mathbf{y} \in \mathbb{R}^n$ if $\mathbf{x} \cdot \mathbf{y} = 0$.

Note that this definition is the same as having angle $\theta = \pi/2$ **except** for the case of the origin which we define to be perpendicular to every other vector. Proceeding still further by analogy with the case of planes in space, we now make a terminology for ‘hyperplanes’ in \mathbb{R}^n .

Definition

If $\mathbf{N} = (N_1, N_2, \dots, N_n)$ is a nonzero vector in \mathbb{R}^n , and if $c \in \mathbb{R}$ is a constant, then the set of solutions $\mathbf{x} \in \mathbb{R}^n$ to the equation

$$\mathbf{N} \cdot \mathbf{x} = c$$

is called a *hyperplane* in \mathbb{R}^n .

Definition

If we write out the equation using coordinates for the point $\mathbf{x} = (x_1, x_2, \dots, x_n)$ on the plane, the equation becomes

$$N_1x_1 + N_2x_2 + \cdots + N_nx_n = c$$

We call \mathbf{N} a *normal vector* to the hyperplane.

We won't go any further with \mathbb{R}^n for now, but it will arise again.