Vectors II

MA1S1

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Anton & Rorres: Ch 3, sec 1 and 2
Hefferon: Ch One, sec II.1 and II.2

Vectors - Review

We introduced the notion of a vector, a quantity with magnitude and direction. Two dimensional vectors can be represented by arrows in the plane.

$\mathbf{v} \quad =$ 

We further defined the components of a vector as the coordinates of the end of the vector when its initial point is at the origin.

$$\mathbf{v} = (v_1, v_2)\ , \text{ also written as } \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Axioms for vectors

A complete set of defining properties for vectors are:

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and a and b are scalars, then

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}-\mathbf{u}=\mathbf{0}$

(v) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$

(vi) $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

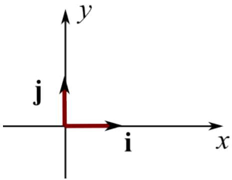
(vii) $a(b\mathbf{u})=(ab)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

These are quite general and make no reference to the dimension of space and we can in fact consider higher dimensional vectors.

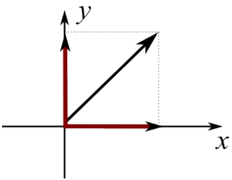
Basis vectors

Another useful way to write a vector is as the sum of basis vectors. We introduce two unit vectors $\mathbf{i}$ and $\mathbf{j}$ pointing along the x- and y-axes respectively.



Now any arbitrary vector can be written as the sum of multiples of these basis vectors. For example

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} \quad =$$



Vectors

- We can add and subtract vectors in component notation: if $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ then

$$\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2) \ .$$

- We can multiply vectors by scalars: if $\mathbf{v} = (v_1, v_2)$ and k is a scalar

$$k\mathbf{v} = (kv_1, kv_2) \ .$$

- How do we write the basis vectors $\mathbf{i}$, $\mathbf{j}$ as 2-tuples?

$$\mathbf{i} \equiv (1, 0) \ , \quad \text{and} \quad \mathbf{j} \equiv (0, 1).$$

- Hence we see

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} \equiv v_1(1, 0) + v_2(0, 1) = (v_1, v_2) \ .$$

- We defined the zero-vector $\mathbf{0} \equiv (0, 0)$ which satisfies

$$0\mathbf{v} = \mathbf{0} \ , \quad \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$$

Vector lengths

We defined the length or norm of a vector $\mathbf{v} = (v_1, v_2)$ as

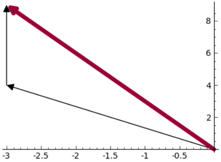
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

Example 1: $\mathbf{v} = (3, 2)$

$$\|v\| = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13} = 3.61...$$

Example 2: if $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = 0\mathbf{i} + 5\mathbf{j}$ what is the length of $\mathbf{v} + \mathbf{w}$?

$$\mathbf{v} + \mathbf{w} = (-3 + 0)\mathbf{i} + (4 + 5)\mathbf{j} \Rightarrow \|\mathbf{v} + \mathbf{w}\| = \sqrt{(-3)^2 + 9^2} = \sqrt{90} \simeq 9.5$$

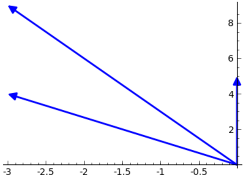


http://www.sagenb.org/home/pub/5036

Vectors

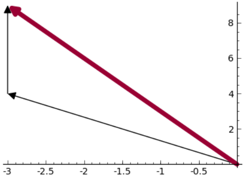
In this note book we will plot the arrows corresponding to the vectors $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i}$. We first plot them in the default color and size.

```
plot(arrow2d((0, 0),(-3,4))+arrow2d((0, 0),(0,5))+arrow2d((0, 0),
(-3,9)),figsize=4)
```



Now we plot them with a variety of colors and sizes: $\mathbf{v}$ and $\mathbf{w}$ as thin black lines and $\mathbf{w} + \mathbf{v}$ as a thick red line. We also shift $\mathbf{w}$ away from the origin so the end coincides with the start of $\mathbf{v}$.

```
plot(arrow2d((0, 0),(-3,4),color=(0,0,0),width=1)+arrow2d((-3, 4),
(-3,9),color=(0,0,0),width=1)+arrow2d((0, 0),(-3,9),color=
(0.6,0,0.2),width=5),figsize=4)
```



Vector lengths

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$$\mathbf{v} + \mathbf{w} = (-3 + 0)\mathbf{i} + (4 + 5)\mathbf{j} \Rightarrow \|\mathbf{v} + \mathbf{w}\| = \sqrt{(-3)^2 + 9^2} = \sqrt{90} \simeq 9.5$$

Note: the length of the sum of vectors is not the sum of lengths
$\|\mathbf{v}\| = \sqrt{25} = 5$, $\|\mathbf{w}\| = \sqrt{25} = 5$, so

$$\|\mathbf{v}\| + \|\mathbf{w}\| = 10 > \|\mathbf{v} + \mathbf{w}\| \simeq 9.5 \ .$$

Dot product

We now define the **dot product** (also know as the **scalar product** or **inner product**) of two vectors $\mathbf{v}$ and $\mathbf{w}$. It is denoted $\mathbf{v} \cdot \mathbf{w}$ and it is a scalar or numerical quantity (**not** another vector). In terms of the components of $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$ we define

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

i.e. the number you get by multiplying the first component of $\mathbf{v}$ by the first component of $\mathbf{w}$, the second by the second **and then adding** these numbers together.

So if (say) $\mathbf{v} = 11\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$, we get

$$\mathbf{v} \cdot \mathbf{w} = 11(3) + (-2)(4) = 25$$

It is really useful to keep in mind that the dot product has a scalar value. Obviously then, if you get a vector answer, it could not possibly be right.

Rules for scalar product

The dot product satisfies some nice algebraic rules. Here are the basic rules, satisfied by any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any scalar k

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(iii) $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (k\mathbf{w})$

(iv) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

You might be tempted to take these for granted but let us check that our definition of the scalar product for 2-tuples satisfy the second rule: $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ so

$$(\mathbf{u} + \mathbf{v}) = (u_1 + v_1, u_2 + v_2) \ ,$$

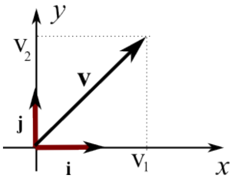
so that

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 \\ &= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 \\ &= (u_1w_1 + u_2w_2) + (v_1w_1 + v_2w_2) \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \ . \end{aligned}$$

Try the other three!

Vectors II - Review

We considered two-dimensional vectors which can be graphically represented as arrows e.g.



Given an coordinate system such vectors can be written as either two tuples $\mathbf{v} = (v_1, v_2)$ or in terms of basis vectors $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.
We defined the scalar (or dot or inner) product of two two-dimensional vectors

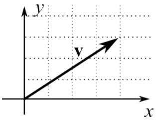
$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \ , \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$$

to be

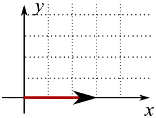
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 \ .$$

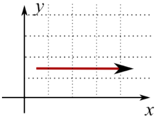
Vectors - Question

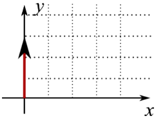
Consider the vector $\mathbf{v} = (v_1, v_2)$



Which diagram shows the vector $\mathbf{w} = v_1\mathbf{i}$?

(A) 

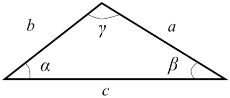
(B) 

(C) 

(D) None of the above.

Geometric scalar product

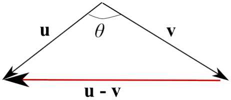
Recall that for a triangle with sides of length a, b, c and angles α, β, γ:



the triangle cosine rule gives the length of the third side in terms of the other two sides and the opposite angle:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Now take two vectors $\mathbf{u}$ and $\mathbf{v}$ and place them end-to-end. Let θ be the angle (such that $0 \leq \theta \leq \pi$ where the angle is measured in **radians** and not degrees[1]).



From the triangle cosine rule we have the length of the vector $\mathbf{u} - \mathbf{v}$

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

[1] 180^o corresponds to π radians

Geometric scalar product

We can also write the length of the vector $\mathbf{u} - \mathbf{v}$ as

$$\begin{aligned} \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}-2\mathbf{u}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v} \end{aligned}$$

Comparing this with the previous result

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

we see that

$$\boxed{\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta\ .}$$

This gives us a trigonometric way to calculate the scalar product. However, we can view it the other way and calculate the angle between two vectors in terms of there components:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{u_1v_1+u_2v_2}{\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}}.$$

An Example

Find the angle between the vectors $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = 5\mathbf{i} + 2\mathbf{j}$
We calculate everything in the formula $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$ except $\cos\theta$.
We get

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{2^2 + (-3)^2} \\
&= \sqrt{13} \\
\|\mathbf{w}\| &= \sqrt{5^2 + 2^2} \\
&= \sqrt{29} \\
\mathbf{v} \cdot \mathbf{w} &= (2)(5) + (-3)(2) \\
&= 10 - 6 \\
&= 4 \\
\Rightarrow \cos\theta &= \frac{4}{\sqrt{13}\sqrt{29}}
\end{aligned}$$

and so $\theta = \mathrm{ArcCos}(0.21) = 1.3633$ (radians).

Orthogonal vectors

We can see from the scalar product formula that if two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, i.e the angle between them is $\pi/2$ and so $\cos \pi/2 = 0$, they have vanishing scalar product: $\mathbf{u} \cdot \mathbf{v} = 0$.

To be more accurate, we can conclude this only when $\mathbf{u}$ and $\mathbf{v}$ are both non-zero vectors. If $\mathbf{u}$ and/or $\mathbf{v}$ is the zero vector then they will have length zero. So that

$$0 = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

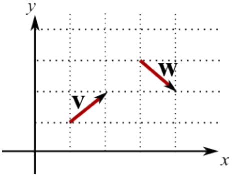
This formula is true regardless of the value of θ as $\|\mathbf{0}\| = 0$. In fact, the problem is that we can't talk about the angle between the zero vector and any other vector because we agreed that the zero vector should have no particular direction.

To get around this we make the *convention* that the zero vector is to be considered perpendicular (or orthogonal is another word for the same idea) to every other vector (including itself!).

With this convention in place we can say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal exactly when $\mathbf{u} \cdot \mathbf{v} = 0$.

Vectors - Question

Consider the vectors $\mathbf{v}$ and $\mathbf{w}$:



What is $\mathbf{v} \cdot \mathbf{w}$?

(A) $\sqrt{2}$

(B) 0

(C) 1

(D) Can't tell from the diagram.

Vectors - Question

Consider the vectors

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Which of the following is **not** a linear combination of $\mathbf{v}$ and $\mathbf{w}$ of the form $\mathbf{v} + t\mathbf{w}$ for some t?

(A) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

(B) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$

(C) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

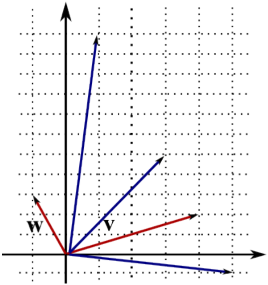
(D) $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$

Vectors

Consider the vectors

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

What does $\mathbf{v} + t\mathbf{w}$ mean geometrically



Vectors

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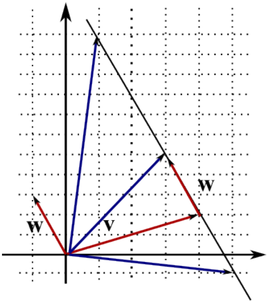


We find all such linear combinations form a line. That is a one-dimensional subspace.

Vectors

Consider the vectors

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Question: Is there any vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

that we can't write as $a\mathbf{v} + b\mathbf{w}$?

A Yes.

B No.

Vectors

Consider the vectors

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Question: Is there any vector

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

that we can't write as $a\mathbf{v} + b\mathbf{w}$?
Answer: Need to solve

$$\begin{aligned} av_1 + bw_1 &= u_1 \\ av_2 + bw_2 &= u_2 \end{aligned}$$

where we have been given $v_1, w_1, u_1, v_2, w_2, u_2$ and we need to find a and b.
Do these equations alway have solutions?

Need to solve

$$\begin{aligned} av_1 + bw_1 &= u_1 \\ av_2 + bw_2 &= u_2. \end{aligned}$$

$$\Rightarrow \qquad a = \frac{w_2 u_1 - u_2 w_1}{v_1 w_2 - v_2 w_1}\ , \qquad b = \frac{v_1 u_2 - v_2 u_1}{v_1 w_2 - v_2 w_1}$$

So there is a solution as long as $v_1 w_2 - v_2 w_1 \neq 0$, that is

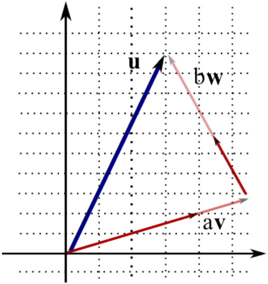
$$\frac{v_1}{v_2} \neq \frac{w_1}{w_2}$$

i.e. the vectors are not parallel.

What about our problem

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

There is a solution as long as $v_1 w_2 - v_2 w_1 \neq 0$.
Check: $4.3 - 2(-1) = 14 \neq 0$



Axioms for vectors

A complete set of defining properties for vectors is:
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(v) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$

(vi) $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

(vii) $a(b\mathbf{u})=(ab)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

and for the scalar product

(a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

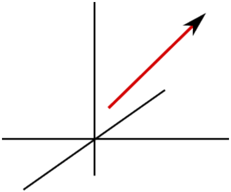
(c) $(a\mathbf{v})\cdot\mathbf{w}=a(\mathbf{v}\cdot\mathbf{w})=\mathbf{v}\cdot(a\mathbf{w})$

(d) $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}\geq 0$, and $\|\mathbf{v}\|^2=0$ if and only if $\mathbf{v}=\mathbf{0}$.

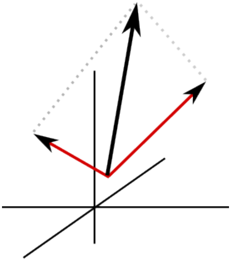
These are quite general and make no reference to the dimension of space and we can in fact consider higher dimensional vectors.

Vectors in space

These same rules work if we consider vectors are geometric arrows living in three-dimensional space.

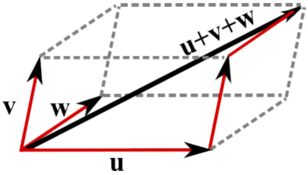


The geometric rules (the triangle rule or the parallelogram rule) for vector addition still hold we now simply think of the vectors as in space, however they still form the edges of a two-dimensional triangle or parallelogram.



Vectors in space

Let us see how the associativity of addition works. We have 3 three-dimensional vectors **u**, **v**, and **w** which we place end to end forming the edges of a parallelepiped

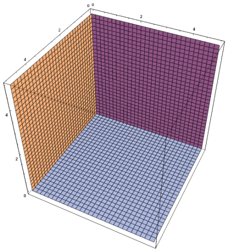


The resultant vector is the diagonal of this three-dimensional volume. As it doesn't matter in what order we go around the edges of the parallelepiped to reach the far corner the geometric picture should make it clear that it doesn't matter in what order we add the vectors i.e.

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Vectors in space

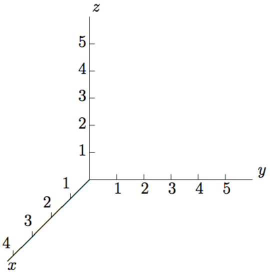
We are used to using two coordinates (x, y) to locate points in a plane (with reference to two fixed perpendicular axes). For points in space we need 3 coordinates and 3 axes. Tricky to draw on the screen!
One way to think about how coordinates work in space is to imagine 3 perpendicular axes meeting at the corner of a room (an ordinary boxy shaped room)



Look at the blue bit as the floor and the brown and purple as walls. So we are looking down and to the left into this corner.
We think of 3 axes meeting at the corner, one along the floor to the left which we will call the x-axis, one along the floor at the back called the y-axis and the third vertical (where the walls meet) and we call that the z-axis.

3-space coordinates

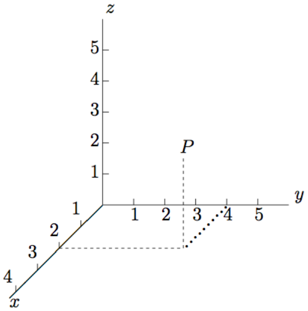
We are more often going to draw just the 3 axes like this



You can see we need to imagine the x-axis as coming out of the picture towards us. And we have a scale on each axis.

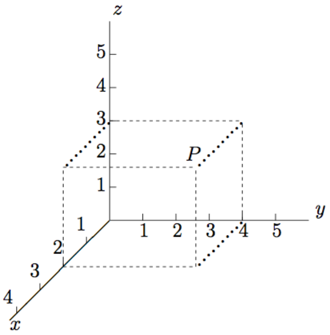
3-space coordinates

To describe where a point is with our room, we do it in two steps. First we describe a point on the floor directly below our point. For this we need just two coordinates (x, y). Here is an example where $x = 2$, $y = 4$ and $z = 3$



3-space coordinates

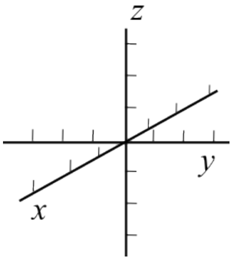
Then we use a third coordinate z to give a height (or altitude) of our point above the floor. For our example $x = 2$, $y = 4$ and $z = 3$ we have



In this picture you can see a box outlined which has its corner at the corner of our “room” and the point P we are describing is at the corner of the box farthest away from where the origin is (which is at the corner of the room).

3-space coordinates

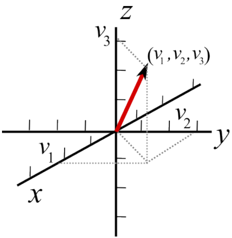
You see that we can describe points in space by 3 coordinates (x, y, z). Given a point you can find coordinates for it, and given coordinates you can locate the point precisely. (All this is supposing you know where your axes are.)



One additional thing to realise is that coordinates can be negative as well as positive. For example, when $z = 0$ we are at height 0, or on the floor, or on the horizontal x-y plane and when $z < 0$ we mean a depth below the floor.

Vector components

Every vector $\mathbf{v}$ can be drawn as an arrow starting from the origin $(0,0,0)$ and it is then uniquely determined it by the coordinates of its terminal point.



We thus associate to each 3d vector a point in a 3d coordinate system. We can then write the vector as a row of three numbers (a row vector) or 3-tuple

$$\mathbf{v} = (v_1, v_2, v_3)$$

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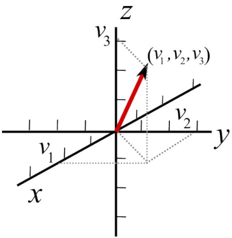

We thus associate to each 3d vector a point in a 3d coordinate system. Or equivalently in the form of a column (a column vector)

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

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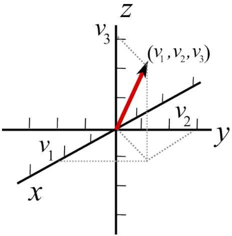
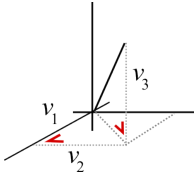


Conversely, to each point, P, in a 3d coordinate system we can associate a vector (its position vector) which is often denoted $\overrightarrow{OP}$, which extends from the origin, O, to the point P.

Vector length

For a vector in 3 dimensions, $\mathbf{v} = (v_1, v_2, v_3)$, you need to use Pythagoras' theorem twice as there are two right-angled triangles.



We know from the one use of the theorem that the length of the diagonal x-y is

$$\sqrt{v_1^2 + v_2^2}\ .$$

We can now use the second right-angled triangle to calculate the length of the vector.

$$\|\mathbf{v}\|^2 = (\sqrt{v_1^2 + v_2^2})^2 + v_3^2$$

In summary then we have

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

(a very similar formula to the familiar one in two dimensions, with just the extra component v_3).

Vectors in space - example

Consider the vectors in three-dimensional space $\mathbf{u} = (2, 3, 4)$ and $\mathbf{w} = (-1, 2, -5)$.
We can easily calculate the length of these vectors using the general formula: for $\mathbf{v} = (v_1, v_2, v_3)$ the norm is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

so that

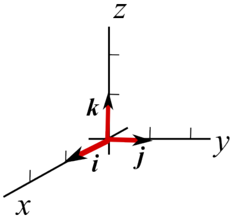
$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \\ \|\mathbf{w}\| &= \sqrt{(-1)^2 + 2^2 + (-5)^2} = \sqrt{30}\ . \end{aligned}$$

The length of the sum is

$$\|\mathbf{u} + \mathbf{w}\| = \sqrt{(2-1)^2 + (3+2)^2 + (4-5)^2} = \sqrt{27} \leq \sqrt{29} + \sqrt{30}$$

Basis vectors in space

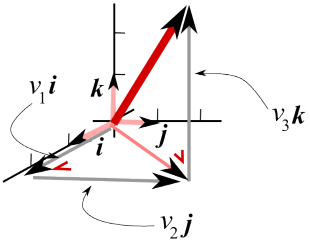
Consider the vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$. These point along the three-coordinate axes of the x-y-z coordinate system in the positive directions and have unit length, $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$.



This is the obvious generalisation of the two basis vectors we had in the plane.

Basis vectors in space

As every vector $\mathbf{v}$ in space can be drawn as an arrow starting from the origin $(0,0,0)$ and its end point at some position, say (v_1, v_2, v_3) we can write it as a linear combination of the basis vectors.

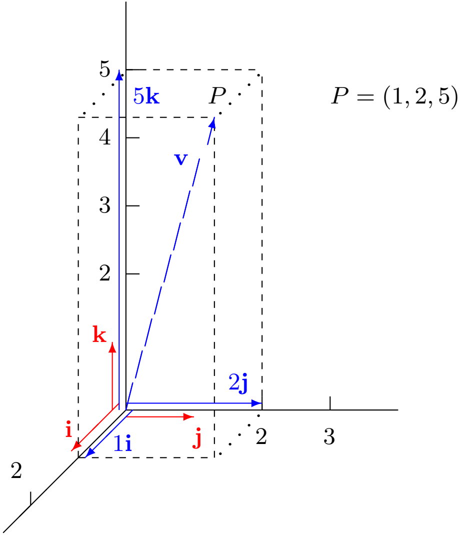


To start we write $\mathbf{v}$ as a sum of a horizontal and a vertical vector parallel to the z-axis. Then we can write the horizontal one (in the horizontal plane) as a sum of two perpendicular (still horizontal) vectors parallel to the x- and y-axes. The end result is that we can write

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\ .$$

The following picture illustrates this when $\mathbf{v}$ ends at $(1, 2, 5)$ and so

$$\mathbf{v} = 1\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$



So we can go from a point $P = (x, y, z)$ to its position vector

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and from a vector back to the point for which it is the position vector.

We can also do computations for e.g. $7\mathbf{v} + 6\mathbf{w}$ with components. It is really no harder than in 2 dimensions although there is one extra component to each vector, and that adds to the amount of arithmetic.