Vectors

MA1S1

Tristan McLoughlin

tristan@maths.tcd.ie

Vectors

Some quantities (which we will call scalars) have purely numerical values (like mass, volume, temperature) while others (which are called vectors) also have a direction associated with them.

Examples of *vector* quantities with a magnitude and direction include:

(i) Wind velocity — which means both the speed (a numerical value) and the direction (like from the West, blowing towards the East);

(ii) Momentum – has a magnitude and a direction.

(iii) Force — will have a strength and a direction in which it acts. This is the rate of change of momentum, changes in vectors are vectors!

An important point is that we don't consider any of these as *fixed* at any particular place.

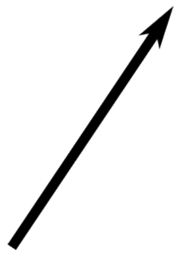
Vectors as arrows

We will denote vectors by boldface letters, $\mathbf{v}, \mathbf{u}, \mathbf{w}$ or by arrows, $\vec{v}, \vec{u}, \vec{w}$ (easier when writing).

Graphically, or geometrically, we think of a vector as an arrow where the length of the arrow represents the magnitude and the arrow points in the direction of the vector.

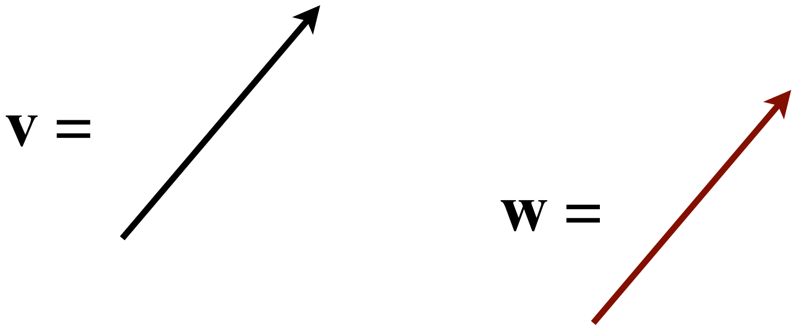
Let's think about two dimensions (or 2-space) first, we can then regard vectors as arrows in a plane.

$\mathbf{v} \quad =$



Vectors as arrows

Consider two vectors, $\mathbf{v}$ and $\mathbf{w}$, with the same length and direction but represented by arrows at different points.



Then we say the vectors are equal

$$\mathbf{v} = \mathbf{w}$$

(because we don't consider vectors as fixed at any position).

Alternatively we can think of the two arrows as two pictorial, or geometric, representations of the same abstract vector.

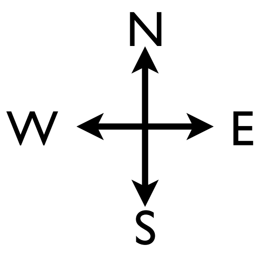
Vectors

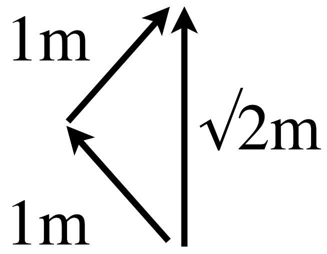
Vectors can be given an abstract and universal definition by the algebraic properties which they satisfy. Geometric vectors give a simple intuitive way to understand these properties.

For example, can we add two vectors?

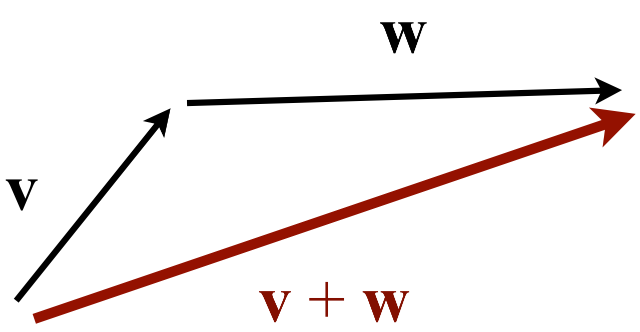
Vector addition

Consider two consecutive displacements (just as in the racetrack game), then the total displacement is the sum of the two individual displacements.





Triangle Rule: If two vectors **v** and **w** are represented by arrows then by placing the arrows tip to end the sum, $\mathbf{v} + \mathbf{w}$, is represented by an arrow from the initial point of the first arrow to the tip of the second.

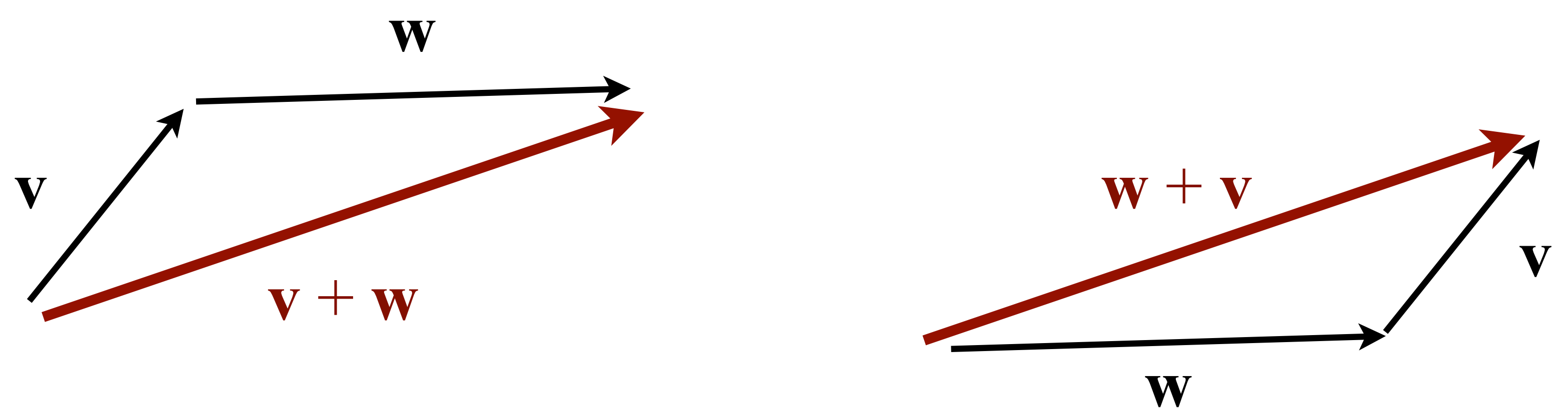


Vector addition

From the rule it should be clear that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

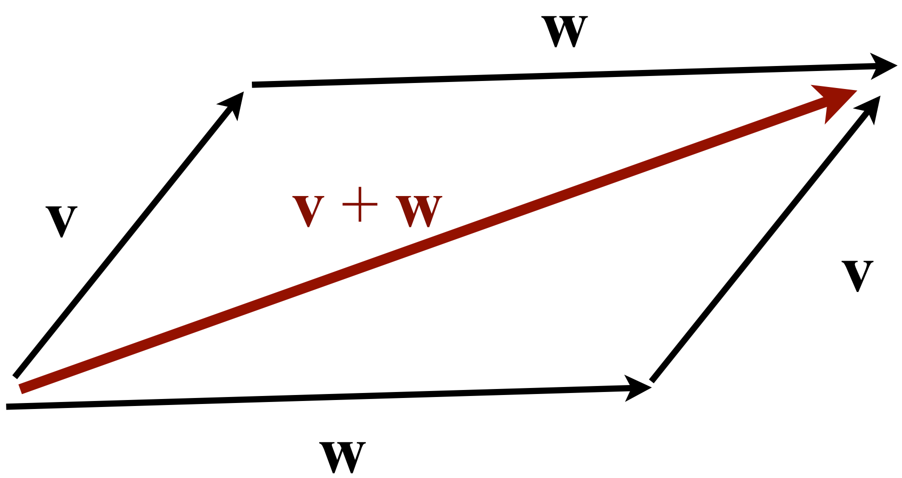
as can be seen from the diagrams



Vector addition

Thus we find

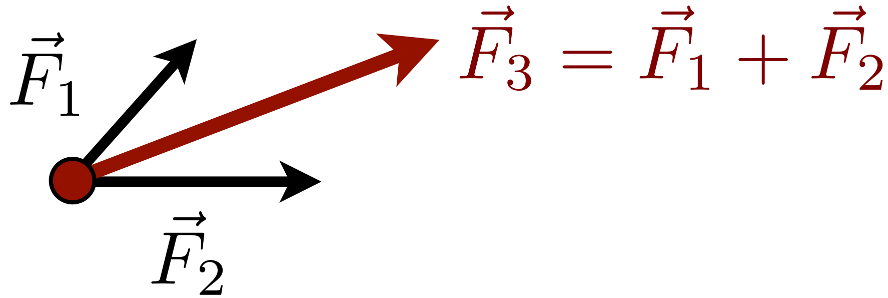
Parallelogram Rule for Vector Addition:
If two vectors $\mathbf{v}$ and $\mathbf{w}$ are represented by arrows then by placing the ends of the arrows together they form the adjacent sides of a parallelogram. The sum, $\mathbf{v} + \mathbf{w}$, is represented by the arrow from the common initial point to the adjacent corner.



$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

Vector addition and physics

While it is a defining property that the sum of two vectors is given by the parallelogram rule, it is an experimental fact that the sum of two forces acting on a particle is given by the same rule.

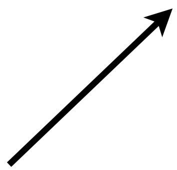
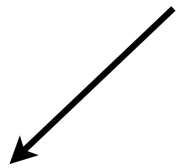


A priori you could imagine $\vec{F}_3 \equiv \vec{F}_3(\vec{F}_1, \vec{F}_2)$ in some complicated fashion. That they add linearly is why linear algebra is so useful in many physical applications.

Negative of a vector

Consider a vector, $\mathbf{v}$, the negative of this vector is a vector with the same magnitude but opposite direction.

Graphically we have:

$\mathbf{v} =$  $\mathbf{w} =$ 

Then we say the vector

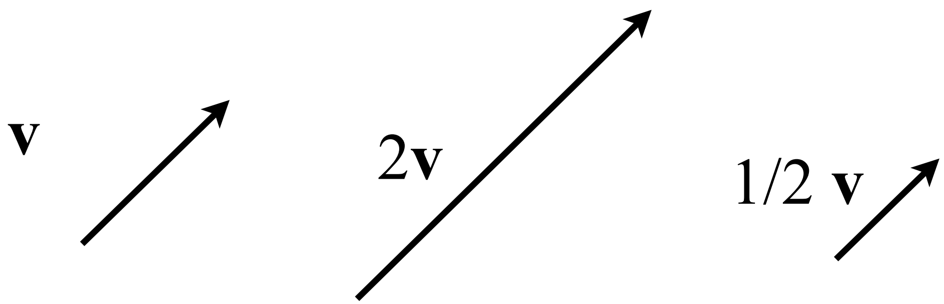
$$\mathbf{v} = -\mathbf{w} = -1(\mathbf{w}) \ .$$

In fact we can multiply a vector by any real number positive or negative.

Multiples of Vectors

Given a vector $\mathbf{v}$ and a number a we can define a new vector $a\mathbf{v}$ which points in the same direction (or opposite direction for negative numbers) and has a magnitude a times larger.

Graphically this means the length is a times the length of the original vector. For example,



Zero vector

It is useful to define the vector $\mathbf{0}$ or $\vec{0}$ as the vector with zero length and arbitrary or undefined direction.

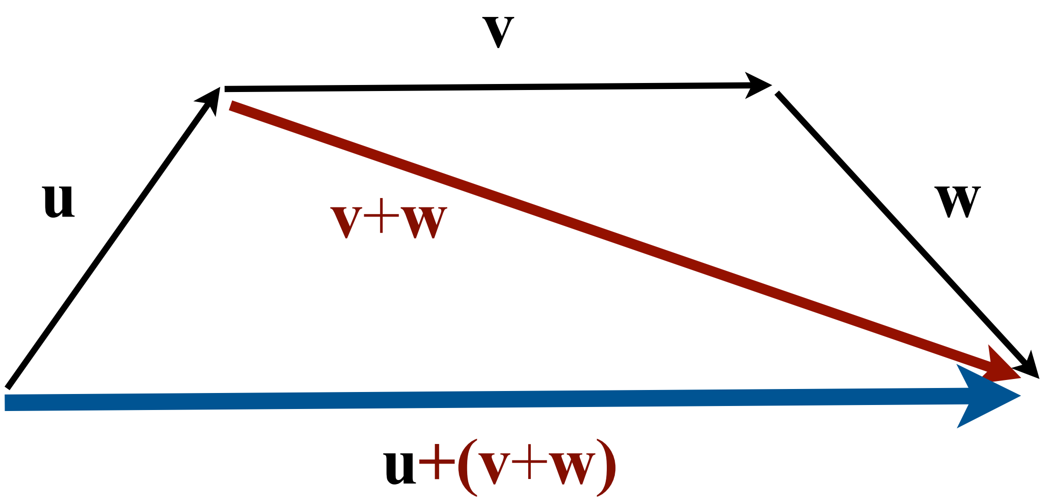
It often arises when vectors cancel each other out. For example if two equal and opposite forces act on a particle the resultant force is the zero vector or when we multiply any vector $\mathbf{v}$ by zero, $0\mathbf{v} = \mathbf{0}$.

Rules for vector addition

There are two more defining properties of vector addition and multiplication by numbers.

- Addition is associative: Given three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can add them in any order

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \ .$$

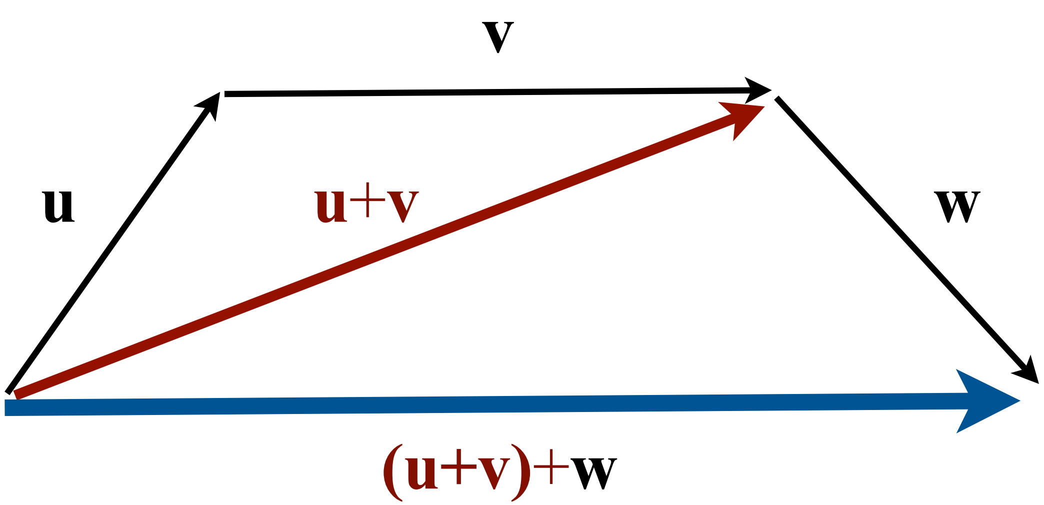


Rules for vector addition

There are two more defining properties of vector addition and multiplication by numbers.

- Addition is associative: Given three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we can add them in any order

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \ .$$

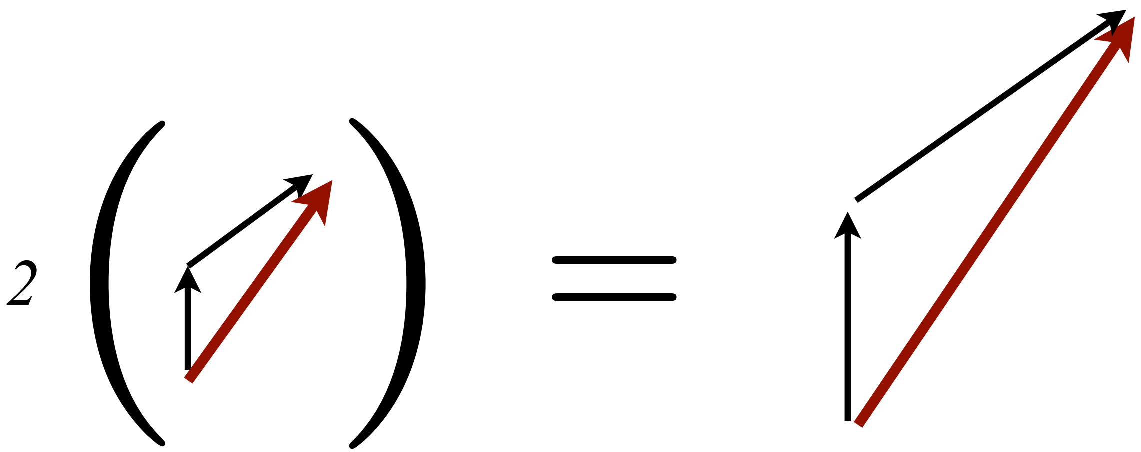


Rules for vector addition

There are two more defining properties of vector addition and multiplication by numbers.

- Multiplication by a number is distributive: Given two vectors, $\mathbf{u}$, $\mathbf{v}$ and a scalar (i.e. a number) a it is equivalent to add the vectors first and then multiply by a or multiply them separately by a and then add them.

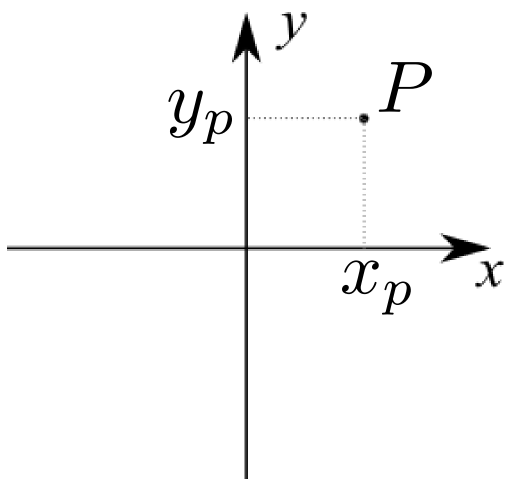
$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \ .$$



Coordinate systems

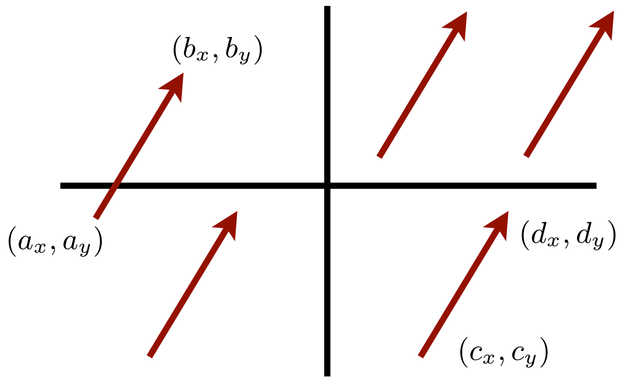
Computations are significantly simplified if we make use of a specific coordinate system (allows the use of algebra!).

A coordinate system in two dimensions is given by two perpendicular axes (usually labelled x and y) which intersect at the origin.



Each point in the plane can be labelled by it's coordinates (x_P, y_P).

Vector Components



All these arrows correspond to the same vector i.e. a vector extending from the point

$$(a_x, a_y) \quad \text{to} \quad (b_x, b_y)$$

is the same as one extending from

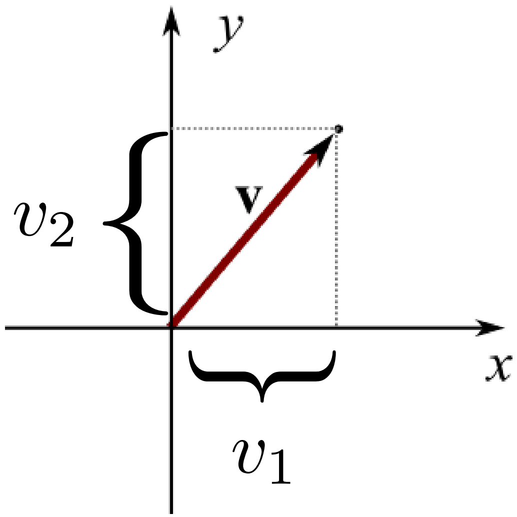
$$(c_x, c_y) \quad \text{to} \quad (d_x, d_y)$$

if $d_x - c_x = b_x - a_x$and $d_y - c_y = b_y - a_y$. We usually describe such a vector as simply

$$(b_x - a_x, b_y - a_y) \quad \text{or} \quad \left(\begin{array}{c} b_x - a_x \\ b_y - a_y \end{array} \right)$$

So "one left and two up" is $(1, 2)$.

If we place our vector, $\mathbf{v}$ at the origin, this is called the canonical (or standard) position, it is completely determined by the coordinates, (v_1, v_2), of its terminal point (tip).



These coordinates are called the components of the $\mathbf{v}$ with respect to the coordinate system and we write

$$\mathbf{v} = (v_1, v_2)$$

Such ordered pairs of numbers are also called 2-tuples. We often just refer to the point (v_1, v_2) as the vector.

Vector length

We will denote the length or magnitude of a vector $\mathbf{v}$ by $||\mathbf{v}||$, this is often also called the norm of the vector.
Given the components, (v_1, v_2), of a vector we can find its length from the usual Pythagorean formula

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$

Note:

- The norm is non-negative.
- Only the zero vector has zero length.
- If $\mathbf{v}$ is any vector and k is any scalar the

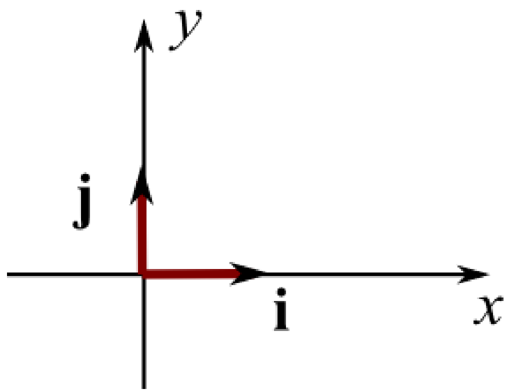
$$||k\mathbf{v}|| = |k| \; ||\mathbf{v}|| \; .$$

- If $\mathbf{v}$ is any vector we can make a vector of unit length, often called a unit vector, $\hat{\mathbf{v}}$ by dividing by the norm

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||} \; , \quad ||\hat{\mathbf{v}}|| = 1.$$

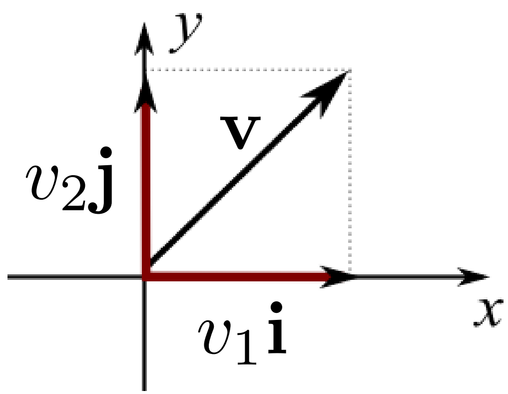
Basis vectors

Another useful way to write a vector is as the sum of basis vectors. We introduce two unit vectors $\mathbf{i}$ and $\mathbf{j}$ pointing along the x- and y-axes respectively.



Now any arbitrary vector can be written as the sum of multiples of these basis vectors. For example

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} \quad =$$



Axioms for vectors

A complete set of defining properties for vectors are:

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and a and b are scalars, then

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}-\mathbf{u}=\mathbf{0}$

(v) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$

(vi) $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

(vii) $a(b\mathbf{u})=(ab)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

These are quite general and make no reference to the dimension of space and we can in fact consider higher dimensional vectors.

Next time: Dot products & Vectors in three and higher dimensions