Graph Theory
MA1S1

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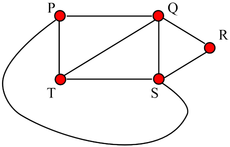
Anton & Rorres: 10.6
Robin J. Wilson: Introduction to Graph theory

Graph Theory

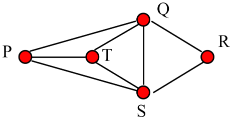
Recall some definitions (actually we will be a bit more general than before):

A graph consists of points called vertices connected by edges.
or
A graph is a pair $(V(G), E(G))$.
$V(G)$ is the vertex set, a non-empty, finite set of elements called vertices.
$E(G)$, the edge set, is a finite family of **un**ordered pairs.

Graphs are equivalent up to redrawings which preserve the connections of edges to vertices

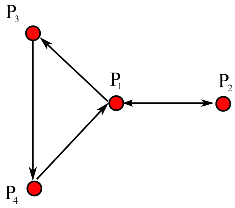


is the same as

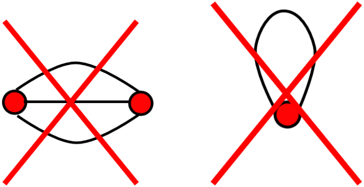


Directed graphs

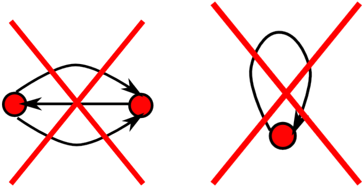
Directed graphs (or digraphs) are graphs where the edges have a direction denoted by an arrow. That is the edge set is made of **ordered** pairs, called arcs.



A simple graph is a graph with no multiple edges or loops. That is nothing like

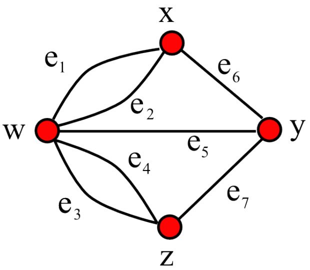


For a simple directed graph we must have distinct arcs. That is no multiple edges pointing the same direction and no loops.



The degree of a vertex is the number of edges which have that vertex as an endpoint.

When two vertices are connected by an edge we say they are adjacent. We say the two vertices are incident on the edge.



For example in this graph all vertices are adjacent. The vertex x is incident upon edges e_1, e_2 and e_6. The vertex y is incident upon edges e_6, e_5 and e_7. The vertex z upon e_7 e_4 and e_3 while w is incident upon e_1, e_2, e_3 and e_4.

More terminology

A walk is a sequence of vertices and connecting edges. For example a walk of length two is $P \to Q \to R$. A walk in which no repeated edge is a trail. A trail which in addition has no repeated vertex is a path. A trail or path that ends at its starting point is closed and a closed path with at least one edge is called a circuit.

A graph in which any two vertices are connected by a path is called a connected graph.

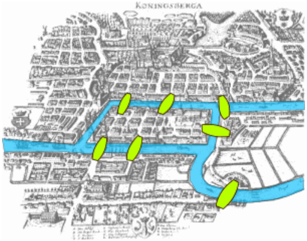
A graph which contain walks that include every edge exactly once is called Eulerian.

In fact Leonhard Euler (1707-1783), a Swiss mathematician

is in many regards the father of graph theory (among many other mathematical achievements).

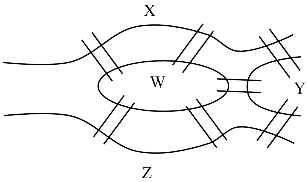
Königsberg Bridge Problem

In the city of Königsberg there were seven bridges, and it was a question as to whether a person could leave their home, cross all the bridges exactly once and return home?

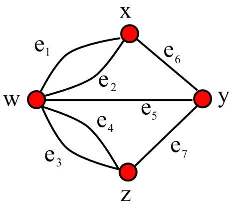


Königsberg Bridge Problem

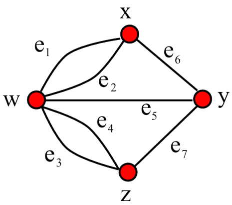
In the city of Königsberg there were seven bridges, and it was a question as to whether a person could leave their home, cross all the bridges exactly once and return home?
If we draw it a bit more schematically



This problem can be mapped to a graph.



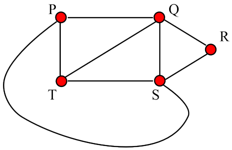
and the problem to whether the graph is Eulerian.



The answer is **no** because of the theorem

Theorem *A connected graph G is Eulerian if and only its vertices all have even degree.*

A Hamiltonian graph is a graph which has a closed walk which contains every **vertex** exactly once. For example:



has the path

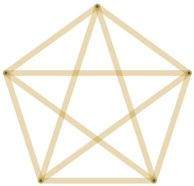
$$P \to Q \to R \to S \to T \to P$$

and so is Hamiltonian. It is however not Eulerian for example

$$P \to Q \to T \to P \to S \to R \to Q \to S \to T$$

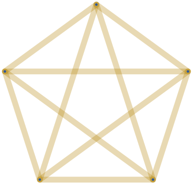
which contains every edge once ends at a different vertex from where it started.

Any graph which can be redrawn on a page so that no edges cross are called planar graphs. Graphs which cannot are called non-planar. For example the graph:

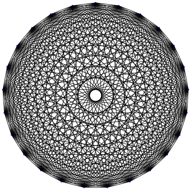


A complete graph is a simple graph in which every pair of distinct vertices are adjacent. The complete graph in n vertices is usually called K_n and has $\frac{1}{2}n(n-1)$ edges.

We've just seen K_5



but also that on the course webpage - K_{25}:



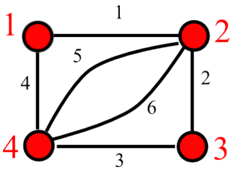
If we add up all the degrees of all the vertices of a graph the result is an even number - twice the number of edges! (Handshaking lemma).

A subgraph of a graph G is a graph all of whose vertices belong to vertex set of G and whose edges belong to the edge set of G.

If G is a graph with a vertex set $\{1, 2, \ldots, n\}$ we define the adjacency matrix M to be the $n \times n$ matrix whose (i, j)-th entry is the number of edges (or arcs for digraphs) joining the i-th and j-th vertex. (i.e. i and j are adjacent).

We can also define the incidence matrix by further numbering the edges $\{1, 2, \ldots, m\}$ and forming the matrix N with entries $n_{ij} = 1$ if vertex i is incident on edge j and zero otherwise.

An example



with the adjacency matrix

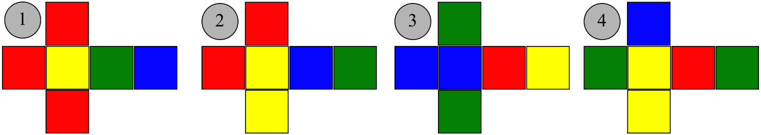
$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and the incidence matrix

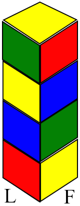
$$N = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

A puzzle

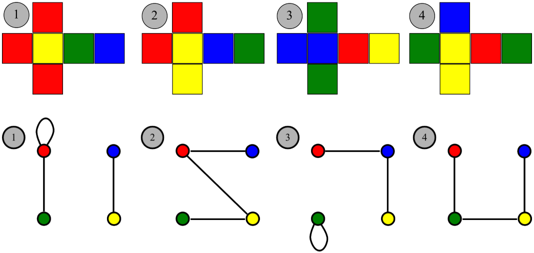
Consider four cubes with sides red, blue, green and yellow:



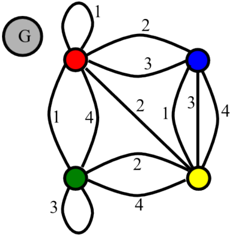
How can we stack these cubes so that each of the resulting four faces shows one of each of the colours?



Think of each cube as a graph on four vertices, one for each colour. Two vertices are adjacent if the colours are on opposite faces:



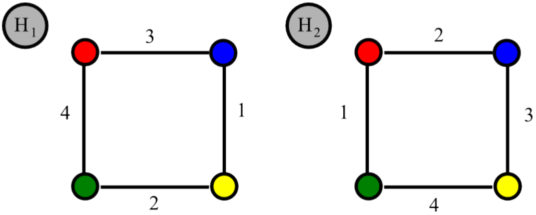
Now we want to combine or superimpose these graphs to form a new graph G.



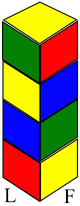
Every solution for the stacked cubes has two faces of each colour on each of the two pairs of opposite sides of the stack (let us call them Front & Back and Left & Right).
Thus the required solution is obtained by finding two subgraphs of G: H_1 for the Front&Back pair and H_2 for the Left&Right pair which satisfy

- have no edges in common
- contain exactly one edge of each cube
- contain only vertices of degree two

In the above example two such subgraphs are



which correspond to the solution



A proof of the theorem for Eulerian graphs

First we prove the following lemma:

Lemma: *If G is a graph in which the degree of every vertex is at least two, then G contains a circuit i.e. a closed walk with distinct edges and vertices.*

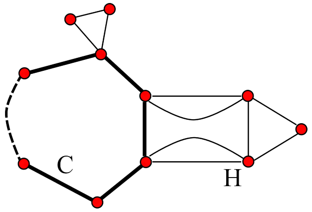
Proof: If G contains any multiple edges or loops the result is trivial so we further suppose that G is a simple graph. Let v be any vertex of G. We can construct a walk $v \to v_1 \to v_2 \to \ldots$ by choosing v_1 to be any vertex adjacent to v, and for subsequent steps by choosing v_{i+1} to be any adjacent vertex to v_i **except** v_{i-1} (that there is such a vertex is guaranteed by the assumption in the lemma). Since G has only finitely many vertices, we will be forced to eventually choose a vertex which has already be chosen, let us say the first such vertex is v_k. Then that part of the walk which lies between occurrences of v_k is a circuit.

A proof of the theorem for Eulerian graphs

Theorem: *A connected graph G is Eulerian if and only if the degree of every vertex of G is even.*

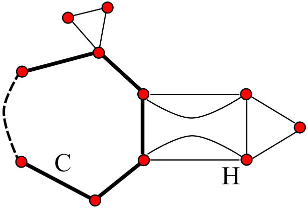
Proof: **First** we prove that Eulerian graphs have even degree vertices: Recall that Eulerian graphs are those which have an Eulerian trail which in turn is a walk which includes each edge exactly once. So suppose P is an Eulerian trail for graph G. Whenever P passes through any vertex, there is a contribution of two toward the degree of that vertex. Since every edge occurs exactly once in P, every vertex must have even degree.

Second we prove that every connected graph with only even degree vertices has an Eulerian trail. Suppose the statement is true for all such graphs with n edges. Let us now consider a graph with $n+1$ edges. Since G is connected every vertex has degree of least two, and so by the previous lemma, G contains a circuit, C:



If the circuit C contains every edge of G, the proof is complete. If not we remove from G the edges of C to a form new (possibly disconnected) graph H which has fewer edges than G and will also have every vertex with even degree. Hence, by assumption, the theorem is true for H (or each of its components) and so each component of H has an Eulerian trail.

We can now contract the required Eulerian trail for G as follows:



By following the edges of C until we reach a vertex belonging to a non-isolated component of H, then tracing the Eulerian trail of the component of H which starts and ends at that vertex, and then continuing along the edges of C until we reach the next component of H and so on. This process terminates when we return to initial vertex of C.

We have now proved that if the theorem holds for all graphs with $n-1$ or fewer edges it holds for all graphs with n edges. One can easily check that it holds for small graphs with say one or two edges. Hence, it holds for 3 edges, then 4 edges and so on for all numbers of edges. We have proven the theorem inductively.