Holographic Mellin Amplitudes in Various Dimensions

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Based on

Workshop on higher-point correlation functions and integrable AdS/CFT
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The story of gluon scattering in flat spacetime

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Maximally Helicity Violating (MHV)

Parke-Taylor Formula:

\[
A_n[1^+ \ldots i^- \ldots j^- \ldots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}
\]

Complicated ways to write zeros!

Amplitudes can be “bootstrapped”:

– dimensional analysis
– Lorenz invariance
– locality
– etc.
Holographic correlators = on-shell scattering amplitudes in a maximally symmetric spacetime

We expect nice properties too:
- Flat space limit.
- $AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4$ SYM should have some hidden simplicity.
Holographic correlators

Infinitely many “particles”: Kaluza–Klein modes from the n–sphere.

1/2 BPS operators:

\[ O_{I_1 \ldots I_k}^{I_k} \Delta = \epsilon k \quad \epsilon = \frac{d}{2} - 1 \]

k–fold symmetric–traceless representation of SO(n+1).

Dual to scalar fields with \( m^2 = \Delta(\Delta - d) \).

The analogue of S–matrix:

\[ \langle O_{k_1}(x_1) \ldots O_{k_n}(x_n) \rangle \]
The standard algorithm: Witten diagram expansion.

n=2,3 is boring, the form is determined by symmetry. Starting from n=4, the dependence on coordinates becomes non-trivial.

At sub-leading order in 1/N,

\[ \langle \mathcal{O}_{k_1}(x_1) \cdots \mathcal{O}_{k_4}(x_4) \rangle = \sum \left[ \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \right] + \left[ \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \right] \]

**External legs:** bulk-to-boundary propagators \( G_{B\partial}^{\Delta_i}(x_i, Z) \)

**Internal legs:** bulk-to-bulk propagators \( G_{BB}^{\Delta, \ell}(Z, W) \)

**Vertices:** expand the effective Lagrangian.

Integrate over the AdS.
Four-point functions

Contact diagrams:

\[ D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \int_{AdS_{d+1}} dZ \ G^{\Delta_1}_{B\partial}(x_1, Z) G^{\Delta_2}_{B\partial}(x_2, Z) G^{\Delta_3}_{B\partial}(x_3, Z) G^{\Delta_4}_{B\partial}(x_4, Z) \]

\[ D_{1111} \] is the scalar one-loop box diagram in four dimensions.

Exchange diagrams:

when the quantum numbers satisfy special relations, an exchange diagram can be expressed as a finite sum of contact Witten diagrams [D’Hoker Freedman Rastelli] e.g., condition in the s-channel

\[ \Delta - \ell = \Delta_1 + \Delta_2 - 2m \quad \text{or} \quad \Delta - \ell = \Delta_3 + \Delta_4 - 2m' \]

for positive integers \( m \) and \( m' \).
Four-point functions

\[
\langle \mathcal{O}_{k_1}(x_1) \ldots \mathcal{O}_{k_4}(x_4) \rangle = \sum_{\Delta_1, \Delta_2, \Delta_3, \Delta_4} \delta^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4} \delta^{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4} \delta^{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4} \delta^{\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4} \delta^{\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4}
\]

A total **nightmare** to implement!

- Exchange diagrams may or may not simplify. The diagrams proliferates quickly with increased external weights.
- Obtaining the vertices are extremely complicated (the 15-page results of Arutynov and Florov for AdS5 quartic vertices).

Only a **handful** of explicit 1/2 BPS correlators over the last 20 years:

- **AdS5**: \(k=2,3,4\) [Arutyunov Frolov, Arutyunov Dolan Osborn Sokatchev, Arutyunov Sokatchev], (higher \(k\) conjecture) [Dolan Nirchl Osborn], near-extremal: \((n+k, n-k, k+2, k+2)\) [Berdichevsky Naaijkens, Uruchurtu].
- **AdS7**: \(k=2\) (stress–tensor multiplet) [Arutynov Sokatchev].
- **AdS4, AdS6**: none.

What is the organizational principle? Where is the hidden simplicity??
Ingredient 1: Mellin representation

Mellin representation of conformal correlators [Mack, Penedones]

\[ G_{\text{conn}}(x_i) = \int [d\delta_{ij}] \prod_{i<j} (x_{ij}^2)^{-\delta_{ij}} M[\delta_{ij}] \]

The integration variables are constrained by

\[ \delta_{ij} = \delta_{ji}, \quad \delta_{ii} = -\Delta_i, \quad \sum_{j=1}^{n} \delta_{ij} = 0 \]

\(M[\delta_{ij}]\) is called the reduced Mellin amplitude. Consider OPE

\[ \mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k c_{ij}^k \left( (x_{ij}^2)^{-\frac{\Delta_i + \Delta_j - \Delta_k}{2}} \mathcal{O}_k(x_k) + \text{descendants} \right) \]

Then \(M[\delta_{ij}]\) should have simple poles at

\[ \delta_{ij} = \frac{\Delta_i + \Delta_j - (\Delta_k + 2n)}{2}, \quad n \geq 0 \]

\[ \tau_k = \Delta_k - \ell_k \]
The constraints are solved by introducing auxiliary “momentum” variables

\[ \delta_{ij} = p_ip_j \]

They satisfy “momentum conservation” and “on–shell” condition

\[ \sum_i p_i = 0 \quad , \quad p_i^2 = -\Delta_i \]

Let us take \( n=4 \), there are two independent variables (“Mandelstam variables”)

\[ s = \Delta_1 + \Delta_2 - 2\delta_{12} \quad , \quad t = \Delta_1 + \Delta_4 - 2\delta_{14} \quad , \quad u = \Delta_1 + \Delta_3 - 2\delta_{13} \]

with the constraint

\[ s + t + u = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 . \]

In the s–channel OPE limit (1,2 come close), a primary operator leads to poles at

\[ s = \tau_\mathcal{O} + 2m \quad , \quad m \geq 0 \quad , \quad \tau_\mathcal{O} = \Delta_\mathcal{O} - \ell \]

Similar statements for the t and u channels.
Further define the Mellin amplitude $\mathcal{M}$ \cite{Mack}

$$M(\delta_{ij}) = \mathcal{M}(\delta_{ij}) \prod^{i<j} \Gamma(\delta_{ij})$$

Two benefits:
- The Gamma factors are such that $\mathcal{M}$ has polynomial residues for conformal blocks.
- At large $N$, the Gamma functions precisely account for the double–trace operators \cite{Penedones}

For example, $\Gamma(\delta_{12})$ has poles that correspond to (recall $\delta_{12} = \frac{\Delta_1 + \Delta_2 - s}{2}$)

$$\mathcal{O}_{\Delta_1} \partial^{\ell} \Box^n \mathcal{O}_{\Delta_2}$$

with twist $\tau = \Delta_1 + \Delta_2 + 2n + O(1/N^2)$.

At large $N$, the Mellin amplitude is meromorphic, with simple poles corresponding to single–trace operators.
Witten diagrams in Mellin space

Contact diagrams: for a vertex with $2k$ derivatives, the Mellin amplitude is a degree–$k$ polynomial [Penedones]

Exchange diagrams (s–channel): simple analytic structure

$$\mathcal{M}(s, t) = \sum_{m=0}^{\infty} \frac{Q_{\ell,m}(t)}{s - \tau - 2m} + P_{\ell-1}(s, t)$$

Same pole and same residue as a conformal block with the same quantum number!

A remarkable simplification: when $\tau + 2m_0 = 2\Delta$, the infinite series truncate into a finite sum. Equivalent to the previous truncation in position space.

It has a very natural explanation in Mellin space.
The truncation of poles must happen in order to be consistent with the 1/N expansion:

– Exchanging an operator contributes $U^{\Delta - \ell} g_{\Delta, \ell}^{\text{coll}}(V) + \ldots$

If there is a small anomalous dimension

$$\Delta \rightarrow \Delta + \gamma \quad U^{\tau/2} \rightarrow U^{\tau/2} + \frac{1}{2} \gamma U^{\tau/2} \log(U) + \frac{1}{8} \gamma^2 U^{\tau/2} \log^2(U) + \ldots$$

– Let’s see how this is reproduced in the inverse Mellin transformation:

$\log(U)$ is produced by a double pole, $\log(U)^2$ is produced by a triple pole, etc.
Witten diagrams in Mellin space

\[
G(x_i) = \frac{1}{x_1^{2\Delta} x_2^{2\Delta}} \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2}} - \Delta M(s, t) \Gamma^2 \left[ \frac{2\Delta - s}{2} \right] \Gamma^2 \left[ \frac{2\Delta - t}{2} \right] \Gamma^2 \left[ \frac{2\Delta - u}{2} \right]
\]

On the other hand, using the counting of 4d N=4 SYM, the tree–level Witten diagrams are of order \(O(1/N^2)\), and the anomalous dimension is also of order \(O(1/N^2)\).

Because \(\log^n(U)\) is multiplied by \(\gamma^n\), only \(n=1\) is allowed at tree level. In terms of the Mellin representation, this means at most double poles are allowed in the integrand.

The truncation must occur: the Gamma functions also contains an infinite series of double poles; if the simple poles in \(M(s, t)\) overlaps with these double poles, they have to terminate.
Witten diagrams in Mellin space

Flat space limit: the asymptotic Mellin amplitude encodes the flat space scattering amplitude [Penedones, Fitzpatrick Kaplan]

\[
T(S_{ij}) = \lim_{R \to \infty} \frac{1}{N} \int_{-i\infty}^{i\infty} d\alpha e^{\alpha} \frac{e^{-\frac{\Delta}{2}}}{\alpha^\frac{d-\sum_i \Delta_i}{2}} \mathcal{M} \left( \delta_{ij} = -\frac{R^2 S_{ij}}{4\alpha}, \Delta_i = Rm_i \right)
\]

If we take all the dimensions fixed (do not scale with R, i.e. \( m_i = 0 \)) and consider four–point amplitudes, the prescription further simplifies into

\[
T(S_{ij}) \propto \lim_{\beta \to \infty} \mathcal{M}(\beta s, \beta t)
\]
Ingredient 2: Superconformal Ward Identity

Some kinematics first:

\[ O_k(x, t) \equiv O_k^{I_1 \ldots I_k}(x) t_{I_1} \ldots t_{I_k}, \quad t^I t_I = 0 \quad \text{(null vectors)} \]

\[ G(x_i, t_i) \equiv \langle O_k(x_1, t_1) O_k(x_2, t_2) O_k(x_3, t_3) O_k(x_4, t_4) \rangle \]

Exploit the conformal and R-symmetry covariance:

\[ G(x_i, t_i) \equiv \left( \frac{t_{12} t_{34}}{x_{12}^2 x_{34}^2} \right)^k \mathcal{G}(U, V; \sigma, \tau) \]

where \( \epsilon = d/2 - 1, \ t_{ij} \equiv t_i \cdot t_j, \ x_{ij} \equiv x_i - x_j. \)

Conformal cross ratios

\[ U = \frac{(x_{12})^2 (x_{34})^2}{(x_{13})^2 (x_{24})^2}, \quad V = \frac{(x_{14})^2 (x_{23})^2}{(x_{13})^2 (x_{24})^2} \]

R-symmetry cross ratios

\[ \sigma = \frac{(t_{13})(t_{24})}{(t_{12})(t_{34})}, \quad \tau = \frac{(t_{14})(t_{23})}{(t_{12})(t_{34})} \]
Superconformal Ward Identity

Now we further impose the constraints from the fermionic generators. We get the superconformal Ward identity [Dolan Gallot Sokatchev]:

\[(\chi \partial_\chi - \epsilon \alpha \partial_\alpha)G(\chi, \chi'; \alpha, \alpha')|_{\alpha=1/\chi} = 0\]

where we have used a change of variables:

\[U = \chi \chi', \quad V = (1 - \chi)(1 - \chi'), \quad \sigma = \alpha \alpha', \quad \tau = (1 - \alpha)(1 - \alpha').\]

R–symmetry group is locally isomorphic to SO(n): 3d OSp(4|N) with even N, 4d (P)SU(2,2|N) with N=2,4, 5d F(4) and 6d OSp(8*|2N) with N=1,2.

In the following we first look at the maximally superconformal cases. **Maximally supersymmetric** theory is very constrained! Non–maximally supersymmetric cases will be discussed in Part II.
Results for maximally SUSY

\[ AdS_4 \times S^7 \]
stress tensor
\[ \langle O_2 O_2 O_2 O_2 \rangle \]

\[ AdS_5 \times S^5 \]
arbitrary KK modes
\[ \langle O_{k_1} O_{k_2} O_{k_3} O_{k_4} \rangle \]

\[ AdS_7 \times S^4 \]
low-lying KK modes
\[ \langle O_k O_k O_k O_k \rangle, \ k = 2, 3, 4 \]
next-next-to-extremal
\[ \langle O_{n+k} O_{n-k} O_{k+2} O_{k+2} \rangle \]

Algebraic bootstrap problem

Mellin superconformal Ward identity

PRL ’17, arXiv: 1710 (L. Rastelli, XZ)

Position space method

arXiv: 1710, 1712.02788 (L. Rastelli, XZ)

arXiv: 1712.02800 (XZ)
Position space method

How it works:

1. Write down an ansatz for the four-point function as a linear combination of all possible exchange Witten diagrams and contact Witten diagrams.

\[ G_{\text{Ansatz}} = \sum_{X \in \{\text{exchanges}\}} \lambda_X X + \sum_{I \in \{\text{contact}\}} c_I I \]

The coefficients $\lambda_X$, $c_I$ could be fixed by using the precise effective Lagrangian, but it’s devilishly complicated. We instead leave the coefficients unfixed.
2. Evaluate all the exchange Witten diagrams as a finite sum of contact diagrams (D–functions).

3. We decompose all the D–functions into a basis spanned by the one–loop box diagram $\Phi(U, V), \log(U), \log(V)$ and 1, with rational function coefficients of $\chi$ and $\chi'$.
   - All the D–functions can be related to $D_{1111}$ by derivative action.
   - $\Phi$ obeys differential recursion relations

\[
\begin{align*}
\partial_{\chi} \Phi &= - \frac{\Phi}{\chi - \chi'} - \frac{\log(V)}{\chi(\chi - \chi')} + \frac{\log(U)}{(-1 + \chi)(\chi - \chi')}, \\
\partial_{\chi'} \Phi &= \frac{\Phi}{\chi - \chi'} + \frac{\log(V)}{\chi'(\chi - \chi')} - \frac{\log(U)}{(-1 + \chi')(\chi - \chi')}.
\end{align*}
\]
4. Impose the superconformal Ward identity. The superconformal Ward identity also has such a unique decomposition.

\[
(\chi \partial_\chi - \epsilon \alpha \partial_\alpha) G_{\text{Ansatz}} \bigg|_{\alpha = 1/\chi} = R_\Phi (\chi, \chi'; \alpha') \Phi + R_{\log U} (\chi, \chi'; \alpha') \log(U)
\]

\[
+ R_{\log V} (\chi, \chi'; \alpha') \log(V) + R_1 (\chi, \chi'; \alpha')
\]

\[
= 0
\]

This becomes a set of linear equations of the unknown coefficients, which turns out to fix all of them up to an overall scaling

\[
R_\Phi (\chi, \chi'; \alpha') = R_{\log U} (\chi, \chi'; \alpha') = R_{\log V} (\chi, \chi'; \alpha') = R_1 (\chi, \chi'; \alpha') = 0
\]

5. The last coefficient can be fixed in terms of a 1/2–BPS three–point function or the free field limit (N=4 SYM).
The position space method improves the traditional method. It circumvents the difficulties in obtaining precise vertices from the effective Lagrangian. But it also has a few obvious shortcomings:

- Needs to evaluate exchange Witten diagrams and express them as finite sum of contact Witten diagrams (complicated).
- Doesn’t work if the spectrum is such that exchange Witten diagrams cannot be expressed as finitely many D–functions. E.g. AdS4.
- The computing time grows with the external conformal dimensions.
- Works in position space and it is not where the true simplicity lies. We should go to Mellin space.
Mellin space method I: an algebraic bootstrap problem

The idea of the method:

1. Solve the superconformal Ward identity in position space first, translate it into an identity in Mellin space. The upshot is a difference operator capturing nontrivial structures in the Mellin amplitude.
2. Combine this structure with other consistency conditions of the Mellin amplitude (Bose symmetry, analytic property, asymptotic behavior) to formulate an algebraic bootstrap problem.
3. We solve it and obtain the general answer.

I will explain this method in the context of N=4 SYM. And for simplicity, I will focus on $\Delta_i = k$ in the following. The method is similar for the general-weight case and I will show the general answer later.
Solution to the superconformal Ward identity

Solution of SCFWI in position space:

\[ \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \mathcal{G}_{\text{free,conn}}(U, V; \sigma, \tau) + R(U, V; \sigma, \tau) \mathcal{H}(U, V; \sigma, \tau) \]

\[ R = (1 - \chi \alpha)(1 - \chi' \alpha')(1 - \chi \alpha')(1 - \chi' \alpha') \]

\[ = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau \]
Solution to the superconformal Ward identity

Solution of SCFWI in position space:

\[ \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \mathcal{G}_{\text{free, conn}}(U, V; \sigma, \tau) + R(U, V; \sigma, \tau) \mathcal{H}(U, V; \sigma, \tau) \]

\[
R = (1 - \chi\alpha)(1 - \chi'\alpha)(1 - \chi\alpha')(1 - \chi'\alpha') \\
= \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma\tau + \tau^2) U + (\sigma^2 - \sigma - \sigma\tau) UV + \sigma V^2 + \sigma\tau
\]
Solution to the superconformal Ward identity

Solution of SCFWI in position space:

\[ G_{\text{conn}}(U, V; \sigma, \tau) = G_{\text{free,conn}}(U, V; \sigma, \tau) + R(U, V; \sigma, \tau) H(U, V; \sigma, \tau) \]

\[ G_{\text{conn}}(U, V; \sigma, \tau) = \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} \mathcal{M}(s, t; \sigma, \tau) \Gamma^2 \left[ \frac{2k - s}{2} \right] \Gamma^2 \left[ \frac{2k - t}{2} \right] \Gamma^2 \left[ \frac{2k - u}{2} \right] \]

\[ R = (1 - \chi \alpha)(1 - \chi' \alpha)(1 - \chi \alpha')(1 - \chi' \alpha') \]

\[ = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau \]
Solution to the superconformal Ward identity

Solution of SCFWI in position space:

\[ \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \mathcal{G}_{\text{free, conn}}(U, V; \sigma, \tau) + R(U, V; \sigma, \tau) \mathcal{H}(U, V; \sigma, \tau) \]

zero Mellin amplitude

\[ \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} \mathcal{M}(s, t; \sigma, \tau) \Gamma^2\left[\frac{2k - s}{2}\right] \Gamma^2\left[\frac{2k - t}{2}\right] \Gamma^2\left[\frac{2k - u}{2}\right] \]

\[ R = (1 - \chi\alpha)(1 - \chi'\alpha)(1 - \chi\alpha')(1 - \chi'\alpha') \]

\[ = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma\tau + \tau^2) U + (\sigma^2 - \sigma - \sigma\tau) UV + \sigma V^2 + \sigma\tau \]
Solution to the superconformal Ward identity

Solution of SCFWI in position space:

\[ \mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \mathcal{G}_{\text{free, conn}}(U, V; \sigma, \tau) + R(U, V; \sigma, \tau) \mathcal{H}(U, V; \sigma, \tau) \]

zero Mellin amplitude

\[
\mathcal{G}_{\text{conn}}(U, V; \sigma, \tau) = \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2}-k} \mathcal{M}(s, t; \sigma, \tau) \Gamma^2\left[\frac{2k-s}{2}\right] \Gamma^2\left[\frac{2k-t}{2}\right] \Gamma^2\left[\frac{2k-u}{2}\right]
\]

\[
\mathcal{H}(U, V; \sigma, \tau) = \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2}-k} \tilde{\mathcal{M}}(s, t; \sigma, \tau) \Gamma^2\left[\frac{2k-s}{2}\right] \Gamma^2\left[\frac{2k-t}{2}\right] \Gamma^2\left[\frac{2k-\tilde{u}}{2}\right]
\]

\[ \tilde{u} = u - 4 \]

\[ R = (1 - \chi \alpha)(1 - \chi' \alpha)(1 - \chi \alpha')(1 - \chi' \alpha') \]

\[ = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau \]
Solution to the superconformal Ward identity

\[ R = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau U^2 \]

Note the multiplication of monomials has the effect of shifting arguments,

\[ U^m V^n \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s, t) = \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s - 2m, t - 2n) \]
Solution to the superconformal Ward identity

$$R = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau U^2$$

Note the multiplication of monomials has the effect of shifting arguments,

$$U^m V^n \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s, t) = \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s - 2m, t - 2n)$$

We hence interpret each monomial as a difference operator

$$\hat{R} = \tau \hat{1} + (1 - \sigma - \tau) \hat{V} + (-\tau - \sigma \tau + \tau^2) \hat{U} + (\sigma^2 - \sigma - \sigma \tau) \hat{U} \hat{V} + \sigma \hat{V}^2 + \sigma \tau \hat{U}^2$$

where each monomial acts as

$$\overleftrightarrow{U^m V^n} \circ \tilde{\mathcal{M}}(s, t; \sigma, \tau) \equiv \tilde{\mathcal{M}}(s - 2m, t - 2n; \sigma, \tau)$$

$$\times \left( \frac{k_1 + k_2 - s}{2} \right)_m \left( \frac{k_3 + k_4 - s}{2} \right)_m \left( \frac{k_2 + k_3 - t}{2} \right)_n$$

$$\times \left( \frac{k_1 + k_4 - t}{2} \right)_n \left( \frac{k_1 + k_3 - u}{2} \right)_{2-m-n} \left( \frac{k_2 + k_4 - u}{2} \right)_{2-m-n}$$
Solution to the superconformal Ward identity

\[ R = \tau 1 + (1 - \sigma - \tau) V + (-\tau - \sigma \tau + \tau^2) U + (\sigma^2 - \sigma - \sigma \tau) UV + \sigma V^2 + \sigma \tau U^2 \]

Note the multiplication of monomials has the effect of shifting arguments,

\[ U^m V^n \int \frac{ds}{2} \frac{dt}{2} U^{s \frac{t}{2}} V^{\frac{t}{2} - k} f(s, t) = \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s - 2m, t - 2n) \]

We hence interpret each monomial as a difference operator

\[ \hat{R} = \tau \hat{1} + (1 - \sigma - \tau) \hat{V} + (-\tau - \sigma \tau + \tau^2) \hat{U} + (\sigma^2 - \sigma - \sigma \tau) \hat{UV} + \sigma \hat{V}^2 + \sigma \tau \hat{U}^2 \]

where each monomial acts as

\[ \overline{U^m V^n} \circ \overline{M}(s, t; \sigma, \tau) \equiv \overline{M}(s - 2m, t - 2n; \sigma, \tau) \]

\[ \times \left( \frac{k_1 + k_2 - s}{2} \right)_m \left( \frac{k_3 + k_4 - s}{2} \right)_m \left( \frac{k_2 + k_3 - t}{2} \right)_n \]

\[ \times \left( \frac{k_1 + k_4 - t}{2} \right)_n \left( \frac{k_1 + k_3 - u}{2} \right)_{2-m-n} \left( \frac{k_2 + k_4 - u}{2} \right)_{2-m-n} \]

A structure of the Mellin amplitude:

\[ \mathcal{M} = \hat{R} \circ \overline{M} \]
Formulating a bootstrap problem

A bootstrap problem

I. superconformal symmetry: \( \mathcal{M} = \tilde{R} \circ \tilde{\mathcal{M}} \)

Other consistency conditions:

– II. Bose symmetry:

\[
\sigma^k \mathcal{M}(u, t; 1/\sigma, \tau/\sigma) = \mathcal{M}(s, t; \sigma, \tau),
\]

\[
\tau^k \mathcal{M}(t, s; \sigma/\tau, 1/\tau) = \mathcal{M}(s, t; \sigma, \tau)
\]

or

\[
\sigma^{k-2} \tilde{\mathcal{M}}(\tilde{u}, t; 1/\sigma, \tau/\sigma) = \tilde{\mathcal{M}}(s, t; \sigma, \tau)
\]

\[
\tau^{k-2} \tilde{\mathcal{M}}(t, s; \sigma/\tau, 1/\tau) = \tilde{\mathcal{M}}(s, t; \sigma, \tau)
\]
Formulating a bootstrap problem

A bootstrap problem

I. superconformal symmetry: $\mathcal{M} = \hat{R} \circ \tilde{\mathcal{M}}$

II. Bose symmetry.

Other consistency conditions:
A bootstrap problem

I. superconformal symmetry: $\mathcal{M} = \hat{R} \circ \tilde{\mathcal{M}}$

II. Bose symmetry.

Other consistency conditions:

– III. **Analytic properties:** the Mellin amplitude has only a **finite** number of simple poles in $s$, $t$, $u$. The positions of the poles are determined by the **twists** of exchanged single-trace operators, which are restricted by **selection rules** of the cubic vertices.

  • R–symmetry selection rules.
  • Twist cut–off.

  \[ \tau < \Delta_1 + \Delta_2 = 2k \]

  Related to the vanishing of extremal cubic coupling in supergravity, also explained by the aforementioned truncation of poles in Mellin space.

  \[ s_0, t_0, u_0 = 2, 4, \ldots, 2k - 2 \]

  The residue at each simple pole is a **polynomial** in the other independent Mandelstam variable.
A bootstrap problem

I. superconformal symmetry: $\mathcal{M} = \widehat{R} \circ \widetilde{\mathcal{M}}$

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III. Analytic properties.

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Other consistency conditions:

- IV. Asymptotics: the growth of the Mellin amplitude at large Mandelstam variables is linear. Required by the good flat space limit.

A spin–J exchange diagram has growth $J-1$, a contact diagram with $2k$ derivatives has growth rate $k$. We have particles with highest spin two. And effectively there are at most two derivatives in the quartic vertices [Arutyunov Frolov, Klabbers, Savin].

$$\mathcal{M}(\beta s, \beta t) \sim \beta^1, \quad \beta \to \infty$$

The correct behavior of flat space scalar amplitude in IIB supergravity.
Formulating a bootstrap problem

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The correct behavior of flat space scalar amplitude in IIB supergravity.

Defines a very constrained bootstrap problem! E.g. we can prove that $\tilde{\mathcal{M}}$ is a rational function.
The solution with equal weights

Solution

Some experimentation leads to

\[ \tilde{\mathcal{M}}(s, t, \tilde{u}; \sigma, \tau) = \sum_{s + J + K = k - 2, 0 \leq I, J, K \leq k - 2} \frac{a_{IJK} \sigma^I \tau^J}{(s - s_M + 2K)(t - t_M + 2J)(\tilde{u} - u_M + 2I)} \]

where \( s_M = t_M = u_M = k - 2 \) and \( \tilde{u} = u - 4 \). The coefficient \( a_{IJK} \) is uniquely solved up to a constant

\[ a_{IJK} = C_{kkk}(k - 2)^2 \]

Reproduces all the position space results: \( k=2,3,4,5 \).

So far we can only prove the uniqueness of the ansatz for \( k=2 \). But we believe the formula holds in general.
The general solution

General solution for arbitrary weights

The solution for arbitrary weights:

$$\tilde{\mathcal{M}}(s, t, \tilde{u}; \sigma, \tau) = \sum_{I + J + K = \mathcal{L} - 2, \ 0 \leq I, J, K \leq \mathcal{L} - 2} \frac{a_{IJK} \sigma^I \tau^J}{(s - s_M + 2K)(t - t_M + 2J)(\tilde{u} - u_M + 2I)}$$

where \( \tilde{u} = u - 4 \) and we assumed \( k_1 \geq k_2 \geq k_3 \geq k_4 \)

$$\mathcal{L} = \min \left\{ k_4, \frac{k_2 + k_3 + k_4 - k_1}{2} \right\}$$

\( s_M = \min\{k_1 + k_2, k_3 + k_4\} - 2 \)

\( t_M = \min\{k_1 + k_4, k_2 + k_3\} - 2 \)

\( u_M = \min\{k_1 + k_3, k_2 + k_4\} - 2 \)

and \( a_{IJK} \) is uniquely determined by the bootstrap conditions.

$$a_{IJK} = \frac{C_{k_1 k_2 k_3 k_4} \left( \frac{\mathcal{L} - 2}{I, J, K} \right)}{(1 + \frac{|k_1 - k_2 + k_3 - k_4|}{2})^I(1 + \frac{|k_1 + k_4 - k_2 - k_3|}{2})^J(1 + \frac{|k_1 + k_2 - k_3 - k_4|}{2})^K}$$

\( C_{k_1 k_2 k_3 k_4} \) can be fixed by taking a lightlike limit in position space. [Aprile, Drummond, Heslop, Paul]
Pros and cons

Pros:

• Turning the task of computing holographic correlators into solving an algebraic problem. Avoid the detour into position space. Never computes a single Witten diagram.
• The difference operator repackages the full Mellin amplitude into some simpler auxiliary amplitude. Interesting structures.
• Similar problem can be set up for AdS7.

Cons:

• Does not work for AdS4. The solution of superconformal Ward identity in 3d involves non-local differential operator, hard to interpret in Mellin space.
• Not always easy to find the right ansatz.
Coffee break!
The solution to the superconformal Ward identity in different spacetime dimensions looks very different. For example, non-local differential operators appear in the solution for odd dimensions (but not in, e.g., 4d).

However the identity itself takes a universal form

\[(\chi \partial_\chi - \epsilon \alpha \partial_\alpha) G(\chi, \chi'; \alpha, \alpha') \big|_{\alpha = 1/\chi} = 0\]

Translate the identity into Mellin space!

Then we obtain the superconformal Ward identity for Mellin amplitudes.
Mellin space superconformal Ward identity

The position space superconformal Ward identity:

\[(\chi \partial \chi - \epsilon \alpha \partial \alpha) G(\chi, \chi'; \alpha, \alpha') \bigg|_{\alpha = 1/\chi} = 0\]

How do we exploit this in Mellin space?

We want

\[U^m V^n \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s, t) = \int \frac{ds}{2} \frac{dt}{2} U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s - 2m, t - 2n)\]

such that we get some difference operators.

But solving \(\chi, \chi'\) in terms of \(U\) and \(V\) leads to nasty square roots due to the asymmetric appearance of \(\chi\) and \(\chi'\).
Mellin space superconformal Ward identity

The position space superconformal Ward identity:

\[
(\chi \partial_\chi - \epsilon \alpha \partial_\alpha) \mathcal{G}(\chi, \chi'; \alpha, \alpha') \bigg|_{\alpha = 1/\chi} = 0
\]

How do we exploit this in Mellin space?

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\[
U^m V^n \int \frac{ds}{2} \frac{dt}{2} \, U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s, t) = \int \frac{ds}{2} \frac{dt}{2} \, U^{\frac{s}{2}} V^{\frac{t}{2} - k} f(s - 2m, t - 2n)
\]

such that we get some difference operators.

But solving \( \chi, \chi' \) in terms of \( U \) and \( V \) leads to nasty square roots due to the asymmetric appearance of \( \chi \) and \( \chi' \).

Make it symmetric!
Mellin space superconformal Ward identity

The position space superconformal Ward identity:

\[(\chi \partial_{\chi} - \epsilon \alpha \partial_{\alpha})G(\chi, \chi'; \alpha, \alpha') \bigg|_{\alpha = 1/\chi} = 0\]

We write

\[\chi \frac{\partial}{\partial \chi} = U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} - \frac{1}{1 - \chi} V \frac{\partial}{\partial V}\]

act with the \(\alpha\) derivatives and set \(\alpha = 1/\chi\).

We bring all the terms to the minimal common denominator. The denominator is \(\chi^L (1 - \chi)\) where \(L\) is the degree of the correlator as a polynomial of the R-symmetry variables.

The numerator is a polynomial in \(\chi\) of degree \(L + 1\)

\[f_0(U, V; \alpha') + \chi f_1(U, V; \alpha') + \chi^2 f_2(U, V; \alpha') + \ldots + \chi^{L+1} f_{L+1}(U, V; \alpha') = 0\]
Mellin space superconformal Ward identity

Notice there is an ambiguity in

\[ U = \chi \chi', \quad V = (1 - \chi)(1 - \chi') , \]

namely, exchanging \( \chi \), \( \chi' \) does not change \( U \) and \( V \). From the previous identity

\[ f_0(U, V; \alpha') + \chi f_1(U, V; \alpha') + \chi^2 f_2(U, V; \alpha') + \ldots + \chi^{L+1} f_{L+1}(U, V; \alpha') = 0 \]

we get another copy for free

\[ f_0(U, V; \alpha') + \chi' f_1(U, V; \alpha') + \chi'^2 f_2(U, V; \alpha') + \ldots + \chi'^{L+1} f_{L+1}(U, V; \alpha') = 0 \]

We add up the two identities and note the following combination

\[ \chi^n + \chi'^n \]

is always a polynomial of \( U \) and \( V \) and we can then interpret them as difference operators.
Mellin space superconformal Ward identity

Procedure of imposing superconformal constraints in Mellin space:

1. Write the derivative of $\chi$ as

$$\chi \frac{\partial}{\partial \chi} = U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} - \frac{1}{1 - \chi} V \frac{\partial}{\partial V}$$

and postpone the action of the U, V derivatives.

2. Decompose the correlator into R–symmetry monomials

$$G(U, V; \sigma, \tau) = \sum_{L+M+N=L} \sigma^M \tau^N G_{LMN}(U, V)$$

and act with the $\alpha$ derivative and set $\alpha = 1/\chi$.

3. Take out a factor $(1 - \chi)^{-1} \chi^{-L}$ to make it a polynomial of $\chi$.

4. Replace $\chi$ with $\chi'$ and get another identity. Add up the two and rewrite all the $\chi$, $\chi'$ as polynomials of U and V.
Procedure of imposing superconformal constraints in Mellin space:

5. We use the Mellin representation of the correlator

\[ G_{LMN}(U, V) = \int_{-i\infty}^{i\infty} \frac{ds}{2} \frac{dt}{2} U^s V^t - \frac{\epsilon(k_3 + k_4)}{2} + \epsilon \mathcal{L} \frac{\epsilon \min\{k_1 + k_4, k_2 + k_3\}}{2} \mathcal{M}_{LMN}(s, t) \Gamma_{k_1 k_2 k_3 k_4} \]

and reinterpret as follows:

\[
U \frac{\partial}{\partial U} \Rightarrow \left[ \frac{s}{2} - \frac{\epsilon(k_3 + k_4)}{2} + \epsilon \mathcal{L} \right] \times \\
V \frac{\partial}{\partial V} \Rightarrow \left[ \frac{t}{2} - \frac{\epsilon \min\{k_1 + k_4, k_2 + k_3\}}{2} \right] \times \\
U^m V^n \Rightarrow \text{shift } s \text{ by } -2m \text{ and shift } t \text{ by } -2n
\]

This procedure works for all the spacetime dimensions.
Next-next-to-extremal correlators in AdS7 (6d (2,0) theories at large c)

\[ k_1 = n + k, \ k_2 = n - k, \ k_3 = k_4 = k + 2 \]

\[ E = k_2 + k_3 + k_4 - k_1 = 4 \quad \text{("extremality")} \]

E=0 and 2 are called extremal and next-to-extremal. E=4 is the first case where the supergravity computation is non-trivial.

Now let’s use the Mellin technique to solve the four-point functions. Analytic structure of the four-point next-next-to-extremal Mellin amplitude is simple: it is composed of a singular part and a regular part. The singular part can be written as a sum of three channels.

\[ M_{\text{ansatz}} = M_s + M_t + M_u + M_c \]

\[ \underbrace{M_s}_{\text{singular}} + \underbrace{M_t}_{\text{singular}} + \underbrace{M_u}_{\text{regular}} + \underbrace{M_c}_{\text{regular}} \]
Moreover the singular part contains only finitely many simple poles, and we know precisely the positions of the poles. They are determined by exchanged single–trace operators subject to the cubic coupling selection rules (R–symmetry and twist cut–off). In s–channel, there are simple poles at \( s=4k+4, 4k+6, \)

\[
\mathcal{M}_s(s, t; \sigma, \tau) = \sum_{0 \leq i, j \leq 2} \sum_{0 \leq a \leq 2} \frac{\lambda_{ij;a}^{(s,1)} \sigma^i \tau^j t^a}{s - (4k + 4)} + \sum_{0 \leq i, j \leq 2} \sum_{0 \leq a \leq 2} \frac{\lambda_{ij;a}^{(s,2)} \sigma^i \tau^j t^a}{s - (4k + 6)}
\]

It has degree–2 in the R–symmetry cross ratios because it’s next–next–to–extremal, and degree–2 in \( t \) because the particles have at most spin 2.

In \( t \) and \( u \)–channels, poles are at \( t=2n, 2n+2, u=2n, 2n+2. \)
The **regular** piece is a **polynomial**, which account for the contact interactions:

\[
\mathcal{M}_c(s, t; \sigma, \tau) = \sum_{0 \leq i, j \leq 2} \sum_{0 \leq a, b \leq 1} \mu_{ij;ab} \sigma^i \tau^j s^a t^b
\]

It is **linear** in \(s\) and \(t\) in order to be consistent with the **flat space limit**: in flat space we have eleven–dimensional supergravity which contains two derivatives.

Such an ansatz is the **most general** ansatz one can write down that is compatible with the qualitative features of the bulk supergravity.
Impose superconformal symmetry using the previous procedure. This is a \textit{finite} problem. All the coefficients are fixed up to an overall normalization.

There is a \textit{hidden simplicity}. In 6d we can also find a difference operator to repackage the full Mellin amplitude into an auxiliary Mellin amplitude. All the next–next–to–extremal auxiliary Mellin amplitude contains one term:

\[
\tilde{\mathcal{M}}(s, t) = \frac{C_{n,k}}{(s - 4k - 6)(s - 4k - 4)(t - 2n - 2)(t - 2n)(\tilde{u} - 2n - 2)(\tilde{u} - 2n)}
\]

Very similar to the case in AdS5.
Applications: AdS4 stress-tensor four-point function

Stress-tensor multiplet four-point function for \( AdS_4 \times S^7 \)

- The dual theory is ABJM at \( k=1 \) with large \( N \).
- We consider 1/2-BPS operators \( O_{k=2} \). It has \( \Delta = 1 \), transforms in the representation \( 35_c \) of SO(8) (or the rank-2 symmetric-traceless representation of \( 8_c \)). It is dual to a scalar in the bulk.
- In the bulk side, there are three kinds of exchange Witten diagrams
  - The scalar field itself.
  - A vector field with \( \Delta = 2 \) and in the representation \( 28 \).
  - A graviton field with \( \Delta = 3 \) and R-symmetry singlet.

All the three fields have twist \( \tau = 1 \). Recall the truncation condition is \( \Delta_1 + \Delta_2 - \tau = 2m \). It implies the series doesn’t truncate and we infinitely many poles.
We will proceed with a simpler ansatz (as in position space approach):

- Use the Mellin amplitudes (singular part) of the exchange Witten diagrams. The singular parts are the same as those of the conformal blocks.
- The ansatz has a singular part which is the sum of exchange amplitudes in three channels, and a regular piece which is a polynomial.

Exchange amplitudes [Mack, Fitzpatrick Kaplan]:

\[
M_{\text{graviton}} = \sum_{n=0}^{\infty} \frac{-3\sqrt{\pi} \cos[n\pi] \Gamma[-\frac{3}{2} - n]}{4n! \Gamma[\frac{1}{2} - n]^2} \frac{4n^2 - 8ns + 8n + 4s^2 + 8st - 20s + 8t^2 - 32t + 35}{s - (2n + 1)}
\]

\[
M_{\text{vector}} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi} \cos[n\pi]}{(1 + 2n) \Gamma[\frac{1}{2} - n] \Gamma[1 + n]} \frac{2t + s - 4}{s - (2n + 1)}
\]

\[
M_{\text{scalar}} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi} \cos[n\pi]}{n! \Gamma[\frac{1}{2} - n]} \frac{1}{s - (2n + 1)}
\]

\[
M_{s\text{-exchange}} = \lambda_g M_{\text{graviton}}(s, t) + \lambda_v (\sigma - \tau) M_{\text{vector}}(s, t) + \lambda_s (4\sigma + 4\tau - 1) M_{\text{scalar}}(s, t)
\]
Applications: AdS4 stress-tensor four-point function

The regular piece is the same as before:

$$\mathcal{M}_c(s, t; \sigma, \tau) = \sum_{0 \leq i, j \leq 2} \sum_{0 \leq a, b \leq 1} \mu_{ij;ab} \sigma^i \tau^j s^a t^b$$

degree–2 in R–symmetry and linear in s and t.

Total ansatz:

$$\mathcal{M}(s, t; \sigma, \tau) = \mathcal{M}_{s\text{-exchange}} + \mathcal{M}_{t\text{-exchange}} + \mathcal{M}_{u\text{-exchange}} + \mathcal{M}_{\text{contact}}$$

The t and u–channel exchange amplitudes are related to the s–channel by crossing symmetry.
Applications: AdS4 stress-tensor four-point function

**Solution:**

The superconformal symmetry fixes all the relative coefficients

\[ \lambda_v = -4\lambda_s, \quad \lambda_g = \frac{\lambda_s}{3} \]

\[ M_{\text{contact}} = \frac{\pi \lambda_s}{2} \left( -s - u\sigma^2 - t\tau^2 + 4(t + u)\sigma\tau + 4(s + u)\sigma + 4(s + t)\tau \right) \]
Applications: AdS4 stress-tensor four-point function

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\[ \lambda_s = -\frac{3\sqrt{2}}{4\pi^2 N^{3/2}} \]

(need a three-point function)
Applications: AdS4 stress-tensor four-point function

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\[ \lambda_s = -\frac{3\sqrt{2}}{4\pi^2 N^{3/2}} \]

(need a three-point function)

Anomalous dimension of \( \mathcal{O}_2^{IJ} \mathcal{O}_2^{IJ} \):

\[ \Delta_0 = 2 - \frac{1120}{\pi^2} \frac{1}{C_T} + \ldots \approx 2 - 113.5 \frac{1}{C_T} + \ldots \]

\( (C_T = \frac{64\sqrt{2}}{3\pi} N^{3/2}) \)

Numerical estimate

\[ \Delta_0^* \approx 2.01 - 109 \frac{1}{C_T} \]

[Agmon, Chester, Pufu]
Applications: AdS4 stress-tensor four-point function

More CFT data can be extracted from the amplitude [Chester] (table on the right). The OPE coefficients match precisely with the localization results and also agree very well with the numerical estimates from the conformal bootstrap.

<table>
<thead>
<tr>
<th>CFT data</th>
<th>ABJ(M) numerical bootstrap</th>
<th>AdS4 Supergravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{(A,2)_1}^{(1)}$</td>
<td>-97</td>
<td>-98.765</td>
</tr>
<tr>
<td>$a_{(A,2)_3}^{(1)}$</td>
<td>-102</td>
<td>-102.045</td>
</tr>
<tr>
<td>$a_{(A,2)_5}^{(1)}$</td>
<td>-104</td>
<td>-103.470</td>
</tr>
<tr>
<td>$a_{(A,+)_0}^{(1)}$</td>
<td>49</td>
<td>48.448</td>
</tr>
<tr>
<td>$a_{(A,+)_2}^{(1)}$</td>
<td>51</td>
<td>51.147</td>
</tr>
<tr>
<td>$a_{(A,+)_4}^{(1)}$</td>
<td>52</td>
<td>52.155</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{0,0,0}}^{(1)}$</td>
<td>-109</td>
<td>-113.480</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{2,0,0}}^{(1)}$</td>
<td>-49</td>
<td>-49.931</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{4,0,0}}^{(1)}$</td>
<td>-33</td>
<td>-33.118</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{6,0,0}}^{(1)}$</td>
<td>-25</td>
<td>-24.935</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{8,0,0}}^{(1)}$</td>
<td>-20</td>
<td>-20.037</td>
</tr>
<tr>
<td>$\Delta_{(A,0)_{10,0,0}}^{(1)}$</td>
<td>-17</td>
<td>-16.762</td>
</tr>
</tbody>
</table>

The technique for imposing superconformal symmetry in Mellin space is also useful beyond the supergravity limit. Together with input from localization, the leading M-theory correction was also recently computed [Chester, Pufu, Yin].
More Applications: SCFTs with eight supercharges

The strategy and techniques also extend to SCFTs with non-maximally superconformal symmetry (eight supercharges in particular). We look at the four-point functions of moment map operators of the flavor current multiplet

\[ \langle O^a(x_1, t_1)O^b(x_2, t_2)O^c(x_3, t_3)O^d(x_4, t_4) \rangle \]

\( O^a_{\alpha\beta}(x) \) has dimension \( \Delta = 2\epsilon \), adjoint of SU(2) R-symmetry and the flavor group.

Used in the superconformal bootstrap program with eight supercharges:

- 4d N=2 [Beem, Lemos, Liendo, Rastelli, van Rees]
- 6d (1,0) [Chang, Lin]
- 5d F(4) [Chang, Fluder, Lin, Wang]
More Applications: SCFTs with eight supercharges

Examples of SCFTs with eight supercharges include:

5d F(4): Seiberg theories $\Rightarrow$ Romans F(4) supergravity coupled to $E_{N_f+1}$ matter

6d (1,0): E-string theory $\Rightarrow$ N=2 gauged supergravity coupled to $E_8$ matter

The method of imposing superconformal constraints is essentially the same. But the computation involves two new features:

1. Only one R-symmetry cross ratio, which makes the superconformal Ward identity weaker.

2. Operators carry flavor symmetry. Each flavor channel gives an identity. All together the constraining power is still pretty strong.
More Applications: SCFTs with eight supercharges

Ansatz:

exchange (flavor current multi.) + exchange (stress tensor multi.) + contact
Ansatz:

\textbf{exchange} (flavor current multi.) + \textbf{exchange} (stress tensor multi.) + \textbf{contact}

Solution:

\[ \lambda_F \] gives the \textbf{superconformal block} of the flavor current multiplet

\[ \lambda_S \] gives the \textbf{superconformal block} of the stress tensor multiplet

\textbf{All} the parameters in the \textbf{contact} part are solved in terms of \( \lambda_S \) and \( \lambda_F \).
More Applications: SCFTs with eight supercharges

An example:
Seiberg theory with $E_8$,

$$248 \otimes 248 = 1 \oplus 3875 \oplus 27000 \oplus 248 \oplus 30380$$

$s$–channel exchange:

$$\mathcal{M}_s^I(s, t; \alpha) = \lambda_{S,g} \mathcal{M}_{\text{graviton}}(s, t) + \lambda_{S,v} P_1(2\alpha - 1) \mathcal{M}_{\text{vector}}(s, t) + \lambda_{S,s} \mathcal{M}_{\text{scalar}}(s, t), \quad \text{if } I = 1,$$

$$\mathcal{M}_s^I(s, t; \alpha) = \lambda_{F,v} \mathcal{M}_{\text{vector}}(s, t) + \lambda_{F,s} P_1(2\alpha - 1) \mathcal{M}_{\text{scalar}}(s, t), \quad \text{if } I = 248 \ (\text{Adj}) ,$$

$$\mathcal{M}_s^I(s, t; \alpha) = 0, \quad \text{otherwise}.$$  

The t–channel and u–channel exchange are related to the s–channel by crossing.

The ansatz for the contact part is linear in $s$ and $t$. 
More Applications: SCFTs with eight supercharges

Superconformal symmetry fixes

\[ \lambda_{S,s} = \lambda_S , \quad \lambda_{S,v} = -\frac{4}{3} \lambda_S , \quad \lambda_{S,g} = \frac{1}{15} \lambda_S , \quad \lambda_{F,s} = \lambda_F , \quad \lambda_{F,v} = -\frac{1}{3} \lambda_F , \]

and

\[
\mathcal{M}_{\text{con}}^{1} = \frac{5\pi \lambda_S (-16350 + 1736s + 3465t)}{47616} - \frac{5\pi \lambda_S (-31146 + 3458s + 3465t)}{23808} - \alpha^2 \frac{5\pi \lambda_S (-20712 + 3451s)}{47616} + \frac{15\pi}{128} (2\alpha^2 - 2\alpha - 1) \lambda_F ,
\]

\[
\mathcal{M}_{\text{con}}^{3875} = -\frac{5\pi \lambda_S (-18 + 7t)}{47616} + \alpha \frac{5\pi \lambda_S (-102 + 14s + 7t)}{23808} - \alpha^2 \frac{5\pi \lambda_S (-40 + 7s)}{15872} + \frac{3\pi}{128} (2\alpha^2 - 2\alpha - 1) \lambda_F ,
\]

\[
\mathcal{M}_{\text{con}}^{27000} = -\frac{5\pi \lambda_S (-18 + 7t)}{47616} + \alpha \frac{5\pi \lambda_S (-102 + 14s + 7t)}{23808} - \alpha^2 \frac{5\pi \lambda_S (-40 + 7s)}{15872} - \frac{\pi}{256} (2\alpha^2 - 2\alpha - 1) \lambda_F ,
\]

\[
\mathcal{M}_{\text{con}}^{248} = \frac{5\pi \lambda_S (-150 + 28s + 21t)}{47616} - \frac{5\pi \lambda_S (18 + 14s - 7t)}{23808} - \alpha^2 \frac{35\pi \lambda_S (-12 + s + 2t)}{47616} + \frac{45\pi}{256} (2\alpha - 1) \lambda_F ,
\]

\[
\mathcal{M}_{\text{con}}^{30380} = \frac{5\pi \lambda_S (-150 + 28s + 21t)}{47616} - \frac{5\pi \lambda_S (18 + 14s - 7t)}{23808} - \alpha^2 \frac{35\pi \lambda_S (-12 + s + 2t)}{47616} .
\]

\[\lambda_S \text{ and } \lambda_F \text{ are fixed by central charges:}\]

\[\lambda_S = -\frac{2480\sqrt{2}}{3\pi^2 N^{5/2}}\]

\[\lambda_F = \frac{240\sqrt{2}}{\pi^2 N^{3/2}}\]
Towards a full-fledged amplitude program in AdS

A lot for the future:

- Make contact with the integrability program.
- Higher KK modes. Is there a simpler auxiliary amplitude for AdS4?
- Loops? BCFT/ICFT?
- More constructive method to reproduce the results: BCFW?
- n-point functions.
- ……
Thank you!