Nested One-point Functions in AdS/dCFT

Georgios Linardopoulos

NCSR "Demokritos" and National & Kapodistrian University of Athens

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based on Phys.Lett. **B781** (2018) 238 [arXiv:1802.01598] and J.Phys. A: Math.Theor. **50** (2017) 254001 [arXiv:1612.06236] with Charlotte Kristjansen and Marius de Leeuw

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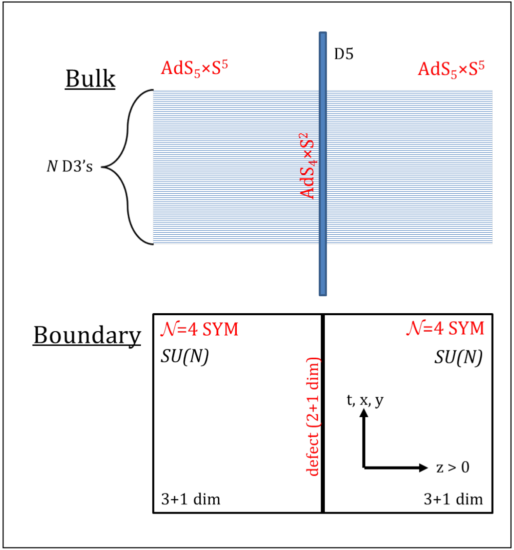
Section 1

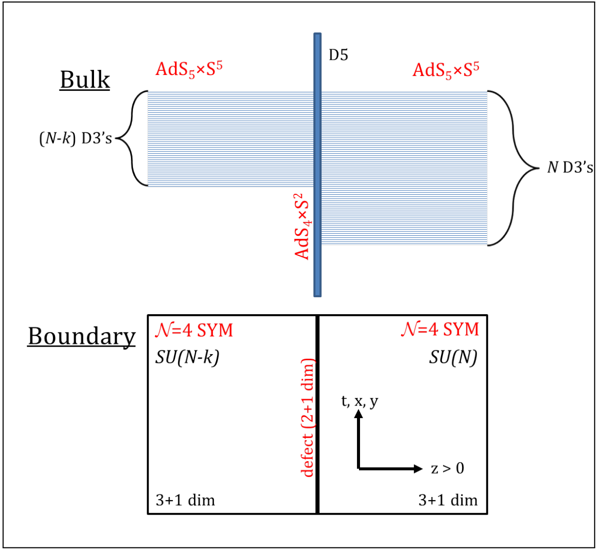
One-point Functions in the D3-D5 System

The D3-D5 system: description



- In the bulk, the D3-D5 system describes IIB Superstring theory on $\text{AdS}_5 \times \text{S}^5$ bisected by D5 branes with worldvolume geometry $\text{AdS}_4 \times \text{S}^2$.
- The dual field theory is still $SU(N)$, $\mathcal{N} = 4$ SYM in $3+1$ dimensions, that now interacts with a SCFT that lives on the 2+1 dimensional defect.
- Due to the presence of the defect, the total bosonic symmetry of the system is reduced from $SO(4,2) \times SO(6)$ to $SO(3,2) \times SO(3) \times SO(3)$.
- The corresponding superalgebra $\mathfrak{psu}(2,2|4)$ becomes $\mathfrak{osp}(4|4)$.

The $(\text{D3-D5})_k$ system



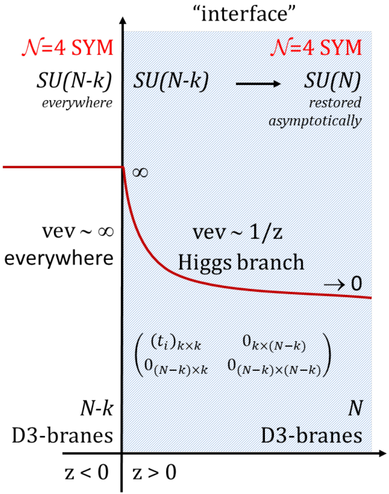
- Add k units of background $U(1)$ flux on the S^2 component of the $AdS_4 \times S^2$ D5-brane.
- Then k of the N D3-branes ($N \gg k$) will end on the D5-brane.
- On the dual SCFT side, the gauge group $SU(N) \times SU(N)$ breaks to $SU(N-k) \times SU(N)$.
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs...

(Karch-Randall, 2001b)

Subsection 2

Nested one-point functions at tree-level

The dCFT interface of D3-D5



- An interface is a wall between two (different/same) QFTs
- It can be described by means of classical solutions that are known as "fuzzy-funnel" solutions (Constable-Myers-Tafjord, 1999 & 2001)
- Here, we need an interface to separate the $SU(N)$ and $SU(N-k)$ regions of the (D3-D5)$_k$ dCFT...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N}=4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\Phi_i}{dz^2} = [\Phi_j, [\Phi_j, \Phi_i]], \quad i,j = 1,\ldots,6.$$

- A manifestly $SO(3) \simeq SU(2)$ symmetric solution is given by $(z > 0)$:

$$\Phi_{2i-1}(z) = \frac{1}{z}\begin{bmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{bmatrix} \quad \& \quad \Phi_{2i} = 0,$$

Nagasaki-Yamaguchi, 2012

where the matrices t_i furnish a k-dimensional representation of $\mathfrak{su}(2)$:

$$[t_i, t_j] = i\epsilon_{ijk} t_k.$$

k-dimensional Representation of $\mathfrak{su}(2)$

We use the following $k \times k$ dimensional representation of $\mathfrak{su}(2)$:

$$t_+ = \sum_{i=1}^{k-1} c_{k,i} E^i_{i+1}, \qquad t_- = \sum_{i=1}^{k-1} c_{k,i} E^{i+1}_i, \qquad t_3 = \sum_{i=1}^{k} d_{k,i} E^i_i$$

$$t_1 = \frac{t_+ + t_-}{2}, \qquad t_2 = \frac{t_+ - t_-}{2i}$$

$$c_{k,i} = \sqrt{i(k-i)}, \qquad d_{k,i} = \frac{1}{2}(k - 2i + 1),$$

where E^i_j are the standard matrix unities that are zero everywhere except (i, j) where they're 1.

1-point functions

Following Nagasaki & Yamaguchi (2012), the 1-point functions of local gauge-invariant scalar operators

$$\langle \mathcal{O}(z, \mathbf{x}) \rangle = \frac{C}{z^{\Delta}}, \qquad z > 0,$$

can be calculated within the D3-D5 dCFT from the corresponding fuzzy-funnel solution, for example:

$$\boxed{\mathcal{O}(z, \mathbf{x}) = \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[\Phi_{2i_1 - 1} \ldots \Phi_{2i_L - 1}\right] \xrightarrow[\text{interface}]{SU(2)} \frac{1}{z^L} \cdot \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[t_{i_1} \ldots t_{i_L}\right]}$$

where $\Psi^{i_1 \ldots i_L}$ is an $\mathfrak{so}(6)$-symmetric tensor and the constant C is given by (MPS=*matrix product state*)

$$C = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \mathrm{MPS} | \Psi \rangle \equiv \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[t_{i_1} \ldots t_{i_L}\right] & (\text{"overlap"}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{i_1 \ldots i_L} \Psi_{i_1 \ldots i_L} & \end{array} \right\},$$

which ensures that the 2-point function will be normalized to unity ($\mathcal{O} \rightarrow (2\pi)^L \cdot \mathcal{O} / (\lambda^{L/2} \sqrt{L})$)

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$$

within $SU(N)$, $\mathcal{N} = 4$ SYM (i.e. without the defect).

Example: chiral primary operators

The one-point functions of the chiral primary operators

$$\mathcal{O}_{\mathrm{CPO}}(x) = \frac{1}{\sqrt{L}} \left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot C^{i_1 \ldots i_L} \operatorname{Tr}\left[\Phi_{i_1} \ldots \Phi_{i_L}\right],$$

where $C^{i_1 \ldots i_L}$ are symmetric & traceless tensors satisfying

$$C^{i_1 \ldots i_L} C^{i_1 \ldots i_L} = 1 \qquad \& \qquad Y_L = C^{i_1 \ldots i_L} \hat{x}_{i_1} \ldots \hat{x}_{i_L}, \qquad \sum_{i=4}^{6} \hat{x}_i^2 = \cos^2 \psi, \qquad \sum_{i=7}^{9} \hat{x}_i^2 = \sin^2 \psi$$

and $Y_L(\psi)$ is the $SO(3) \times SO(3) \subseteq SO(6)$ spherical harmonic, have been calculated at weak coupling:

$$\langle \mathcal{O}_{\mathrm{CPO}}(x) \rangle = \frac{1}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} k \left(k^2 - 1\right)^{L/2} \frac{Y_L(\pi/2)}{z^L}, \qquad k \ll N \to \infty.$$

Nagasaki-Yamaguchi, 2012

The large-k limit agrees with the supergravity calculation at tree-level:

$$\langle \mathcal{O}_{\mathrm{CPO}}(x) \rangle = \frac{k^{L+1}}{\sqrt{L}} \left(\frac{2\pi^2}{\lambda}\right)^{L/2} \frac{Y_L(\pi/2)}{z^L} \cdot \left[1 + \frac{\lambda \, \mathrm{I}_1}{\pi^2 k^2} + \ldots\right], \qquad \mathrm{I}_1 \equiv \frac{3}{2} + \frac{(L-2)(L-3)}{4(L-1)}.$$

Dilatation operator

The mixing of single-trace operators $\mathcal{O}(x)$ is generally described by the integrable $\mathfrak{so}(6)$ spin chain:

$$\mathbb{D} = L \cdot \mathbb{I} + \frac{\lambda}{8\pi^2} \cdot \mathbb{H} + \sum_{n=2}^{\infty} \lambda^n \cdot \mathbb{D}_n, \qquad \mathbb{H} = \sum_{j=1}^{L} \left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1} + \frac{1}{2} \mathbb{K}_{j,j+1} \right), \qquad \lambda = g_{\text{YM}}^2 N,$$

Minahan-Zarembo, 2002

up to one loop in $\mathcal{N} = 4$ SYM, where

$$\mathbb{I} \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = |\ldots \Phi_a \Phi_b \ldots\rangle$$

$$\mathbb{P} \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = |\ldots \Phi_b \Phi_a \ldots\rangle$$

$$\mathbb{K} \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = \delta_{ab} \sum_{c=1}^{6} |\ldots \Phi_c \Phi_c \ldots\rangle \,.$$

The above result is unaffected by the presence of a defect in the SCFT (DeWolfe-Mann, 2004).

Bethe eigenstates

- In the following we will examine eigenstates of the $\mathfrak{so}(6)$ spin chain which can be written as:

$$|\Psi\rangle \equiv \sum_{x_i} \psi_i(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \cdot |\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\downarrow} \bullet \ldots \bullet \underset{x_3}{\Uparrow} \bullet \ldots \bullet \underset{x_4}{\Downarrow} \bullet \ldots\rangle,$$

where $\mathbf{u}_{1,2,3}$ are the rapidities of the excitations at x_i. The corresponding single-trace operator is

$$|\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\downarrow} \bullet \ldots \bullet \underset{x_3}{\Uparrow} \bullet \ldots \bullet \underset{x_4}{\Downarrow} \ldots\rangle \sim \mathrm{Tr}\left[\mathcal{Z}^{x_1-1}\mathcal{W}\mathcal{Z}^{x_2-x_1-1}\mathcal{Y}\mathcal{Z}^{x_3-x_2-1}\overline{\mathcal{W}}\mathcal{Z}^{x_4-x_3-1}\overline{\mathcal{Y}}\ldots\right],$$

where $\mathcal{Z}$ (ground state field), $\mathcal{W}$, $\mathcal{Y}$ (excitations) are the following three complex scalars:

$$\mathcal{W} = \Phi_1 + i\Phi_2 \sim \uparrow \qquad \mathcal{Y} = \Phi_3 + i\Phi_4 \sim \downarrow \qquad \mathcal{Z} = \Phi_5 + i\Phi_6 \sim \bullet$$

$$\overline{\mathcal{W}} = \Phi_1 - i\Phi_2 \sim \Uparrow \qquad \overline{\mathcal{Y}} = \Phi_3 - i\Phi_4 \sim \Downarrow \qquad \overline{\mathcal{Z}} = \Phi_5 - i\Phi_6 \sim \circ$$

- The wavefunction $\psi(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ can be constructed with the (nested) coordinate Bethe ansatz...

Nesting

- Let us first construct the kets $|\bullet\ldots\bullet\underset{x_1}{\uparrow}\bullet\ldots\bullet\underset{x_2}{\downarrow}\bullet\ldots\bullet\underset{x_3}{\Uparrow}\bullet\ldots\bullet\underset{x_4}{\Downarrow}\bullet\ldots\rangle$.

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- Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

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- Start from a closed $\mathfrak{so}(6)$ spin chain of length L:

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- Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".
- Start from a closed $\mathfrak{so}(6)$ spin chain of length L. Excite exactly N_1 sites of the chain:

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Now take the N_1 excitations to be the ground state.

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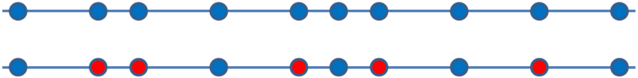
Now take the N_1 excitations to be the ground state. Excite N_2 sites of the new chain...



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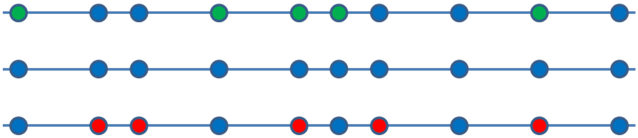
Now take the N_1 excitations to be the ground state. Excite N_2 sites of the new chain... or N_3 sites:



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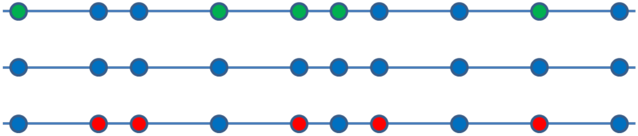

- We end up with three sets/levels of rapidities, one rapidity for each excitation:

$$\mathbf{u}_1 = \{u_{1,j}\}_{j=1}^{N_1}, \qquad \mathbf{u}_2 = \{u_{2,j}\}_{j=1}^{N_2}, \qquad \mathbf{u}_3 = \{u_{3,j}\}_{j=1}^{N_3},$$

each set corresponds to a simple root $\alpha_{1,2,3}$ of $\mathfrak{so}(6)$.

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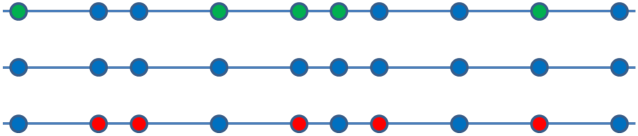

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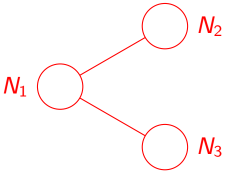
  each set corresponds to a simple root $\alpha_{1,2,3}$ of $\mathfrak{so}(6)$.
- To construct the kets, we must map the sets of rapidities to the available complex scalar fields.

Rapidities & fields

- As we've just seen, each set of rapidities can be associated to a node of the $\mathfrak{so}(6)$ Dynkin diagram:



$$(0 \leq N_1 \leq L,\ 0 \leq N_2 \leq N_1/2,\ 0 \leq N_3 \leq N_2)\,.$$

- The total weight of the $\mathfrak{so}(6)$ representation will then be given by:

$$\mathbf{w} = L\mathbf{q} - N_1\boldsymbol{\alpha}_1 - N_2\boldsymbol{\alpha}_2 - N_3\boldsymbol{\alpha}_3$$

where $\mathbf{q} \equiv (1,0,0)$ and the $\mathfrak{so}(6)$ roots are $\boldsymbol{\alpha}_1 \equiv (1,-1,0)$, $\boldsymbol{\alpha}_2 \equiv (0,1,-1)$, $\boldsymbol{\alpha}_3 \equiv (0,1,1)$.

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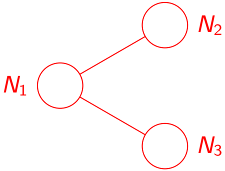

$$\left(0 \le N_1 \le L,\ 0 \le N_2 \le N_1/2,\ 0 \le N_3 \le N_2\right).$$

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- The corresponding Cartan charges are given by:

$$\mathbf{w} = (J_1, J_2, J_3) = (L - N_1, N_1 - N_2 - N_3, N_2 - N_3), \qquad J_1 \ge J_2 \ge J_3 \ge 0.$$

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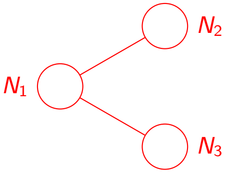

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- Here are the corresponding Dynkin indices:

$$[\mathbf{w}\cdot\boldsymbol{\alpha}_2, \mathbf{w}\cdot\boldsymbol{\alpha}_1, \mathbf{w}\cdot\boldsymbol{\alpha}_3] = [J_2 - J_3, J_1 - J_2, J_2 + J_3] = [N_1 - 2N_2, L - 2N_1 + N_2 + N_3, N_1 - 2N_3]\,.$$

Rapidities & fields

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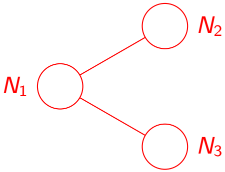

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- Each complex scalar field is associated to the following set of weights:

$$\begin{aligned} \mathcal{Z} &\sim \mathbf{q} & \mathcal{W} &\sim \mathbf{q} - \boldsymbol{\alpha}_1 & \mathcal{Y} &\sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 \\ \overline{\mathcal{Z}} &\sim \mathbf{q} - 2\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 & \overline{\mathcal{W}} &\sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 & \overline{\mathcal{Y}} &\sim \mathbf{q} - \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_3 \end{aligned}$$

Nested Bethe Ansatz

Here's the nested $\mathfrak{so}(6)$ wavefunction (in a somewhat simplified form):

$$\psi_i(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)=\sum_{P_1}A_1(P_1)\prod_{j=1}^{N_1}\frac{1}{u_{1,P_{1,j}}-i/2}\left(\frac{u_{1,P_{1,j}}+i/2}{u_{1,P_{1,j}}-i/2}\right)^{n_{1,j}-1}\cdot\psi_{(2,i)}(\mathbf{u}_1,\mathbf{u}_2)\cdot\psi_{(3,i)}(\mathbf{u}_1,\mathbf{u}_3)$$

where

$$\psi_{(a,i)}(\mathbf{u}_1,\mathbf{u}_a)=\sum_{P_a}A_a(P_a)\prod_{j=1}^{N_a}\frac{1}{u_{a,P_{a,j}}-u_{1,P_{1,n_{a,j}}}-i/2}\prod_{k=1}^{n_{a,j}-1}\frac{u_{a,P_{a,j}}-u_{1,P_{1,k}}+i/2}{u_{a,P_{a,j}}-u_{1,P_{1,k}}-i/2},\qquad a=2,3,$$

and

$$A_a(\ldots,k,j,\ldots)=A_a(\ldots,j,k,\ldots)\,S_a(u_{a,k},u_{a,j}),\quad S_a(u_{a,k},u_{a,j})\equiv\frac{u_{a,k}-u_{a,j}+i}{u_{a,k}-u_{a,j}-i}.$$

Bethe equations

- The periodicity of the Bethe wavefunction ψ (at each nesting level) leads to the Bethe equations:

$$\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L = \prod_{j\neq i}^{N_1} \frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i} \prod_{k=1}^{N_2} \frac{u_{1,i}-u_{2,k}-i/2}{u_{1,i}-u_{2,k}+i/2} \prod_{l=1}^{N_3} \frac{u_{1,i}-u_{3,l}-i/2}{u_{1,i}-u_{3,l}+i/2}, \qquad i=1,\ldots,N_1\equiv M$$

$$1 = \prod_{l\neq i}^{N_2} \frac{u_{2,i}-u_{2,l}+i}{u_{2,i}-u_{2,l}-i} \prod_{k=1}^{N_1} \frac{u_{2,i}-u_{1,k}-i/2}{u_{2,i}-u_{1,k}+i/2}, \qquad i=1,\ldots,N_2\equiv N_+$$

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which must be satisfied by the rapidities of the excitations/Bethe roots.

- Because of the cyclicity of the trace, the momentum carrying roots obey the following relation:

$$\prod_{i=1}^{N_1} \frac{u_{1,i}+i/2}{u_{1,i}-i/2} = 1 \Leftrightarrow \sum_{i=1}^{N_1} p_{1,i} = 0 \qquad \text{(momentum conservation)}.$$

Bethe state overlaps

- The matrix product state projects the 3 complex scalars on the $SU(2)$ fuzzy funnel solution:

$$\boxed{\langle \text{MPS}|\Psi\rangle = z^L \cdot \sum_{1\leq x_k\leq L} \psi(x_k)\cdot \text{Tr}\left[\mathcal{Z}^{x_1-1}\mathcal{W}\mathcal{Z}^{x_2-x_1-1}\mathcal{Y}\mathcal{Z}^{x_3-x_2-1}\overline{\mathcal{W}}\mathcal{Z}^{x_4-x_3-1}\overline{\mathcal{Y}}\dots\right]}$$

where the complex scalar fields $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$ are expressed in terms of the $\mathfrak{su}(2)$ matrices as follows:

$$\mathcal{W} = \overline{\mathcal{W}} = \frac{t_1}{z}, \qquad \mathcal{Y} = \overline{\mathcal{Y}} = \frac{t_2}{z}, \qquad \mathcal{Z} = \overline{\mathcal{Z}} = \frac{t_3}{z}$$

- The corresponding matrix product state (MPS) is given by:

$$|\text{MPS}\rangle = \text{Tr}_a\left[\prod_{\text{l}=1}^{L} |\mathcal{Z}\rangle_{\text{l}} \otimes t_3 + |\mathcal{W}\rangle_{\text{l}} \otimes t_1 + |\mathcal{Y}\rangle_{\text{l}} \otimes t_2 + \text{c.c.}\right].$$

The $\mathfrak{su}(2)$ subsector

For example, let us first consider the subsector that contains only two complex scalars:

$$\mathcal{W} = \Phi_1 + i\Phi_2 \quad \longleftrightarrow \quad |\uparrow\rangle \sim t_1$$

$$\mathcal{Z} = \Phi_5 + i\Phi_6 \quad \longleftrightarrow \quad |\bullet\rangle \sim t_3.$$

This is also known as the $\mathfrak{su}(2)$ subsector of the dCFT. In the $\mathfrak{su}(2)$ subsector, the trace operator $\mathbb{K}_{j,j+1}$ does not contribute to the mixing matrix $\mathbb{D}$:

$$\mathbb{H}_{\mathfrak{su}(2)} = \sum_{j=1}^{L} \left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1}\right).$$

This is just the Hamiltonian of the Heisenberg $\text{XXX}_{1/2}$ spin chain. The MPS can be written as follows:

$$|\text{MPS}\rangle = \text{Tr}_a \left[\prod_{j=1}^{L} \Big(|\uparrow_j\rangle \otimes t_1 + |\bullet_j\rangle \otimes t_3\Big)\right],$$

and it corresponds to the above choice of fields.

$\mathfrak{su}(2)$ Bethe states

In the $\mathfrak{su}(2)$ subsector, $|\Psi\rangle$ is just the coordinate Bethe state $|\mathbf{p}\rangle$:

$$|\mathbf{p}\rangle = \mathfrak{N} \cdot \sum_{\sigma \in S_M} \sum_{1 \leq n_1 \leq \ldots \leq n_M \leq L} \exp\left[i \sum_k p_{\sigma(k)} n_k + \frac{i}{2} \sum_{j<k} \theta_{\sigma(j)\sigma(k)}\right] |\mathbf{x}\rangle, \qquad |\mathbf{p}\rangle \equiv |p_1, p_2, \ldots, p_M\rangle \,.$$

where

$$|\mathbf{x}\rangle \equiv |x_1, x_2, \ldots, x_M\rangle \equiv |\bullet \ldots \bullet \underset{x_1}{\uparrow} \bullet \ldots \bullet \underset{x_2}{\uparrow} \bullet \ldots \bullet \underset{x_M}{\uparrow} \bullet \ldots \bullet\rangle = S^-_{n_1} \ldots S^-_{n_M} |0\rangle$$

and the vacuum state $|0\rangle$ and the raising and lowering operators $S^{\pm}$ have been defined as

$$|0\rangle = \bigotimes_{i=1}^{L} |\bullet\rangle \,, \qquad S^+ |\uparrow\rangle = |\bullet\rangle \quad \& \quad S^- |\bullet\rangle = |\uparrow\rangle \,.$$

The matrix θ_{jk} and the normalization constant $\mathfrak{N}$ are given by:

$$e^{i\theta_{jk}} = \frac{u_j - u_k + i}{u_j - u_k - i} \equiv S_{jk}, \quad u_j \equiv \frac{1}{2} \cot \frac{p_j}{2}, \qquad \mathfrak{N} \equiv \exp\left[-\frac{i}{2} \sum_{j<k} \theta_{jk}\right].$$

The $\mathfrak{su}(3)$ and $\mathfrak{so}(6)$ subsectors

- In the $\mathfrak{su}(3)$ subsector all the three real complex scalars contribute:

$$\mathcal{W} = \Phi_1 + i\Phi_2 \sim t_1, \qquad \mathcal{Y} = \Phi_3 + i\Phi_4 \sim t_2, \qquad \mathcal{Z} = \Phi_5 + i\Phi_6 \sim t_3.$$

The corresponding wavefunction is constructed by means of the nested coordinate Bethe ansatz:

$$\psi = \sum_{P_1, P_2} A_1(P_1) A_2(P_2) \prod_{j=1}^{N_1} \prod_{j=1}^{N_2} \left(\frac{u_{1,P_{1,j}} + i/2}{u_{1,P_{1,j}} - i/2}\right)^{n_{1,j}} \prod_{k=1}^{n_{2,j}} \frac{\left(u_{2,P_{2,j}} - u_{1,P_{1,k}} + i/2\right)^{\delta_{k\neq n_{2,j}}}}{u_{2,P_{2,j}} - u_{1,P_{1,k}} - i/2}$$

$$A_a(\ldots, k, j, \ldots) = A_a(\ldots, j, k, \ldots)\, S_a(u_{a,k}, u_{a,j}), \qquad S_a(u_{a,k}, u_{a,j}) \equiv \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i}.$$

- In the $\mathfrak{so}(6)$ subsector all the three real complex scalars contribute:

$$\mathcal{W} = \overline{\mathcal{W}} = \Phi_1 + i\Phi_2 \sim t_1, \qquad \mathcal{Y} = \overline{\mathcal{Y}} = \Phi_3 + i\Phi_4 \sim t_2, \qquad \mathcal{Z} = \overline{\mathcal{Z}} = \Phi_5 + i\Phi_6 \sim t_3$$

and similarly the $\mathfrak{so}(6)$ wavefunction can be constructed by the nested Bethe ansatz.

Subsection 4

Determinant formulas

M. de Leeuw, C. Kristjansen, G. Linardopoulos, *Scalar One-point functions and matrix product states of AdS/dCFT*. Phys.Lett. **B781** (2018) 238, [arXiv:1802.01598]

1-point functions in $\mathfrak{su}(2)$

In the $\mathfrak{su}(2)$ sector our goal is to calculate the one-point function coefficient:

$$C = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \text{MPS}|\mathbf{p}\rangle}{\langle \mathbf{p}|\mathbf{p}\rangle^{\frac{1}{2}}}, \qquad k \ll N \to \infty.$$

where the $k \times k$ matrices $t_{1,3}$ form a k-dimensional representation of $\mathfrak{su}(2)$:

$$\langle \text{MPS}|\mathbf{p}\rangle = \mathfrak{N} \cdot \sum_{\sigma \in S_M} \sum_{1 \leq x_k \leq L} \exp\left[i \sum_k p_{\sigma(k)} x_k + \frac{i}{2} \sum_{j<k} \theta_{\sigma(j)\sigma(k)}\right] \cdot \text{Tr}\left[t_3^{x_1-1} t_1 t_3^{x_2-x_1-1} \ldots\right].$$

Overlap properties:

- The overlap $\langle \text{MPS}|\mathbf{p}\rangle$ vanishes if $M \equiv N_1$ or L is odd: $\text{Tr}\left[t_3^{x_1-1} t_1 t_3^{x_2-x_1-1} \ldots\right]\Big|_{M \text{ or } L \text{ odd}} = 0$
- The overlap $\langle \text{MPS}|\mathbf{p}\rangle$ vanishes if $\sum p_i \neq 0$: due to trace cyclicity
- The overlap $\langle \text{MPS}|\mathbf{p}\rangle$ vanishes if momenta are not fully balanced $(p_i, -p_i)$: due to $Q_3 \cdot |\text{MPS}\rangle = 0$

de Leeuw-Kristjansen-Zarembo, 2015

The $\mathfrak{su}(2)$ determinant formula

Vacuum overlap:

$$\langle \text{MPS}|0\rangle = \text{Tr}\left[t_3^L\right] = \zeta\left(-L, \frac{1-k}{2}\right) - \zeta\left(-L, \frac{1+k}{2}\right), \qquad \zeta(s, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where $\zeta(s, a)$ is the Hurwitz zeta function. For M balanced excitations the overlap becomes:

$$C_k(\{u_j\}) \equiv \frac{\langle \text{MPS}|\{u_j\}\rangle_k}{\sqrt{\langle \{u_j\}|\{u_j\}\rangle}} = C_2(\{u_j\}) \cdot \sum_{j=(1-k)/2}^{(k-1)/2} j^L \left[\prod_{l=1}^{M/2} \frac{u_l^2\left(u_l^2 + k^2/4\right)}{\left[u_l^2 + (j-1/2)^2\right]\left[u_l^2 + (j+1/2)^2\right]}\right]$$

where

$$C_2(\{u_j\}) \equiv \frac{\langle \text{MPS}|\{u_j\}\rangle_{k=2}}{\sqrt{\langle \{u_j\}|\{u_j\}\rangle}} = \left[\prod_{j=1}^{M/2} \frac{u_j^2 + 1/4}{u_j^2} \frac{\det G^+}{\det G^-}\right]^{\frac{1}{2}},$$

and the $M/2 \times M/2$ matrices $G_{jk}^{\pm}$ and $K_{jk}^{\pm}$ are defined as:

$$G_{jk}^{\pm} = \left(\frac{L}{u_j^2 + 1/4} - \sum_n K_{jn}^{+}\right)\delta_{jk} + K_{jk}^{\pm} \qquad \& \qquad K_{jk}^{\pm} = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}.$$

Buhl-Mortensen, de Leeuw, Kristjansen, Zarembo, 2015

The $\mathfrak{su}(3)$ determinant formula

Moving to the $\mathfrak{su}(3)$ sector, let us define the following Baxter functions Q and R :

$$Q_1(x) = \prod_{i=1}^{M} (x - u_i), \qquad Q_2(x) = \prod_{i=1}^{N_+} (x - v_i), \qquad R_2(x) = \prod_{i=1}^{2\lfloor N_+/2 \rfloor} (x - v_i).$$

All the one-point functions in the $\mathfrak{su}(3)$ sector are then given by

$$C_k(\{u_j; v_j\}) = T_{k-1}(0) \cdot \sqrt{\frac{Q_1(0)\, Q_1(i/2)}{R_2(0)\, R_2(i/2)} \cdot \frac{\det G^+}{\det G^-}}$$

de Leeuw-Kristjansen-GL, 2018

where $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$ and

$$T_n(x) = \sum_{a=-n/2}^{n/2} (x + ia)^L \frac{Q_1(x + i(n+1)/2)Q_2(x + ia)}{Q_1(x + i(a + 1/2))Q_1(x + i(a - 1/2))}.$$

The validity of the $\mathfrak{su}(3)$ formula has been checked numerically for a plethora of $\mathfrak{su}(3)$ states.

The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k(\{u_j\}) = \left[Q_1(0)\, Q_1(i/2) \cdot \frac{\det G^+}{\det G^-}\right]^{1/2} \cdot \sum_{a=(1-k)/2}^{(k-1)/2} \frac{a^L Q_1(ik/2)}{Q_1(i(a+1/2))Q_1(i(a-1/2))},$$

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k(\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1(i/2)}{Q_1(0)} \frac{Q_2^2(i/2)}{R_2(0)\, R_2(i/2)} \cdot \frac{\det G^+}{\det G^-}},$$

de Leeuw-Kristjansen-Mori, 2016

where,

$$\phi_{1,i} = -i \log\left[\left(\frac{u_{1,i} - i/2}{u_{1,i} + i/2}\right)^L \prod_{j\neq i}^{N_1} \frac{u_{1,i} - u_{1,j} + i}{u_{1,i} - u_{1,j} - i} \prod_{k=1}^{N_2} \frac{u_{1,i} - u_{2,k} - \frac{i}{2}}{u_{1,i} - u_{2,k} + \frac{i}{2}} \prod_{l=1}^{N_3} \frac{u_{1,i} - u_{3,l} - \frac{i}{2}}{u_{1,i} - u_{3,l} + \frac{i}{2}}\right]$$

$$\phi_{2,i} = -i \log\left[\prod_{l\neq i}^{N_2} \frac{u_{2,i} - u_{2,l} + i}{u_{2,i} - u_{2,l} - i} \prod_{k=1}^{N_1} \frac{u_{2,i} - u_{1,k} - \frac{i}{2}}{u_{2,i} - u_{1,k} + \frac{i}{2}}\right].$$

The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k(\{u_j\}) = \left[Q_1(0)\, Q_1(i/2) \cdot \frac{\det G^+}{\det G^-}\right]^{1/2} \cdot \sum_{a=(1-k)/2}^{(k-1)/2} \frac{a^L Q_1(ik/2)}{Q_1(i(a+1/2))Q_1(i(a-1/2))},$$

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k(\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1(i/2)}{Q_1(0)} \frac{Q_2^2(i/2)}{R_2(0)\, R_2(i/2)} \cdot \frac{\det G^+}{\det G^-}}.$$

de Leeuw-Kristjansen-Mori, 2016

For $A_\pm = A_1 \pm A_2$, $B_\pm = B_1 \pm B_2$, $C_\pm = C_1 \pm C_2$, we define:

$$G \equiv \frac{\partial \phi_I}{\partial u_J} = \begin{bmatrix} A_1 & A_2 & B_1 & B_2 & D_1 \\ A_2 & A_1 & B_2 & B_1 & D_1 \\ B_1^t & B_2^t & C_1 & C_2 & D_2 \\ B_2^t & B_1^t & C_2 & C_1 & D_2 \\ D_1^t & D_1^t & D_2^t & D_2^t & D_3 \end{bmatrix}, \quad G^+ = \begin{pmatrix} A_+ & B_+ & D_1 \\ B_+^t & C_+ & D_2 \\ 2D_1^t & 2D_2^t & D_3 \end{pmatrix}, \quad G^- = \begin{pmatrix} A_- & B_- \\ B_-^t & C_- \end{pmatrix}.$$

The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k(\{u_j\}) = \left[Q_1(0)\, Q_1(i/2) \cdot \frac{\det G^+}{\det G^-}\right]^{1/2} \cdot \sum_{a=(1-k)/2}^{(k-1)/2} \frac{a^L Q_1(ik/2)}{Q_1(i(a+1/2))Q_1(i(a-1/2))},$$

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k(\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1(i/2)}{Q_1(0)} \frac{Q_2^2(i/2)}{R_2(0)\, R_2(i/2)} \cdot \frac{\det G^+}{\det G^-}}.$$

de Leeuw-Kristjansen-Mori, 2016

Here are some more properties of one-point functions in $\mathfrak{su}(3)$:

- One-point functions vanish if M or $L + N_+$ is odd.
- Because $Q_3 \cdot |\text{MPS}\rangle = 0$ all 1-point functions vanish unless all the Bethe roots are fully balanced:

$$\{u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0\}, \qquad \{v_1, \ldots, v_{N_+/2}, -v_1, \ldots, -v_{N_+/2}, 0\}.$$

The $\mathfrak{so}(6)$ determinant formula

The one-point function in the $\mathfrak{so}(6)$ sector is given by

$$C_k(\{u_j; v_j; w_j\}) = \mathbb{T}_{k-1}(0) \cdot \sqrt{\frac{Q_1(0)\, Q_1(i/2)\, Q_1(ik/2)\, Q_1(ik/2)}{R_2(0)\, R_2(i/2)\, R_3(0)\, R_3(i/2)} \cdot \frac{\det G^+}{\det G^-}}$$

where $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$, $w_k \equiv u_{3,k}$ and

$$\mathbb{T}_n(x) = \sum_{a=-n/2}^{n/2} (x+ia)^L \frac{Q_2(x+ia)\, Q_3(x+ia)}{Q_1(x+i(a+1/2))\, Q_1(x+i(a-1/2))}.$$

de Leeuw-Kristjansen-GL, 2018

More properties of one-point functions in $\mathfrak{so}(6)$:

- One-point functions vanish if M or $L + N_+ + N_-$ is odd.
- Because $Q_3 \cdot |\mathrm{MPS}\rangle = 0$, all 1-point functions vanish unless all the Bethe roots are fully balanced:

$$\{u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0\}$$

$$\{v_1, \ldots, v_{N_+/2}, -v_1, \ldots, -v_{N_+/2}, 0\}, \qquad \{w_1, \ldots, w_{N_-/2}, -w_1, \ldots, -w_{N_-/2}, 0\}.$$

The $\mathfrak{so}(6)$ determinant formula

The norm matrix is defined as follows:

$$G \equiv \partial_J \phi_I = \frac{\partial \phi_I}{\partial u_J} = \begin{bmatrix} A_1 & A_2 & B_1 & B_2 & D_1 & F_1 & F_2 & H_1 \\ A_2 & A_1 & B_2 & B_1 & D_1 & F_2 & F_1 & H_1 \\ B_1^t & B_2^t & C_1 & C_2 & D_2 & K_1 & K_2 & H_2 \\ B_2^t & B_1^t & C_2 & C_1 & D_2 & K_2 & K_1 & H_2 \\ D_1^t & D_1^t & D_2^t & D_2^t & D_3 & D_4^t & D_4^t & H_3 \\ F_1^t & F_2^t & K_1^t & K_2^t & D_4 & L_1 & L_2 & H_4 \\ F_2^t & F_1^t & K_2^t & K_1^t & D_4 & L_2 & L_1 & H_4 \\ H_1^t & H_1^t & H_2^t & H_2^t & H_3^t & H_4^t & H_4^t & H_5 \end{bmatrix},$$

where

$$\phi_I \equiv \{\phi_{1,i}, \phi_{2,j}, \phi_{3,k}\}, \qquad i = 1, \ldots, N_1, \quad j = 1, \ldots, N_2, \quad k = 1, \ldots, N_3$$

$$u_J \equiv \{u_{1,i}, u_{2,j}, u_{3,k}\}, \qquad I, J = 1, \ldots, N_1 + N_2 + N_3,$$

and

$$\phi_{1,i} = -i \log \left[\left(\frac{u_{1,i} - i/2}{u_{1,i} + i/2} \right)^L \prod_{j \neq i}^{N_1} \frac{u_{1,i} - u_{1,j} + i}{u_{1,i} - u_{1,j} - i} \prod_{k=1}^{N_2} \frac{u_{1,i} - u_{2,k} - \frac{i}{2}}{u_{1,i} - u_{2,k} + \frac{i}{2}} \prod_{l=1}^{N_3} \frac{u_{1,i} - u_{3,l} - \frac{i}{2}}{u_{1,i} - u_{3,l} + \frac{i}{2}} \right]$$

$$\phi_{2,i} = -i \log \left[\prod_{l \neq i}^{N_2} \frac{u_{2,i} - u_{2,l} + i}{u_{2,i} - u_{2,l} - i} \prod_{k=1}^{N_1} \frac{u_{2,i} - u_{1,k} - \frac{i}{2}}{u_{2,i} - u_{1,k} + \frac{i}{2}} \right], \quad \phi_{3,i} = -i \log \left[\prod_{l \neq i}^{N_3} \frac{u_{3,i} - u_{3,l} + i}{u_{3,i} - u_{3,l} - i} \prod_{k=1}^{N_1} \frac{u_{3,i} - u_{1,k} - \frac{i}{2}}{u_{3,i} - u_{1,k} + \frac{i}{2}} \right].$$

The $\mathfrak{so}(6)$ determinant formula

- It can be shown that the determinant of the norm matrix factorizes:

$$\det G = \det G_+ \cdot \det G_- ,$$

with $A_\pm \equiv A_1 \pm A_2$ (and so on for $B_\pm$, $C_\pm$, $F_\pm$, $K_\pm$, $L_\pm$), while

$$G_+ = \begin{pmatrix} A_+ & B_+ & D_1 & F_+ & H_1 \\ B_+^t & C_+ & D_2 & K_+ & H_2 \\ 2D_1^t & 2D_2^t & D_3 & 2D_4^t & H_3 \\ F_+^t & K_+^t & D_4 & L_+ & H_4 \\ 2H_1^t & 2H_2^t & 2H_3^t & 2H_4^t & H_5 \end{pmatrix} \quad \& \quad G_- = \begin{pmatrix} A_- & B_- & F_- \\ B_-^t & C_- & K_- \\ F_-^t & K_-^t & L_- \end{pmatrix}.$$

- An unproven claim (Escobedo, 2012) is that the norm of any $\mathfrak{so}(6)$ Bethe eigenstate is given by the determinant of its norm matrix:

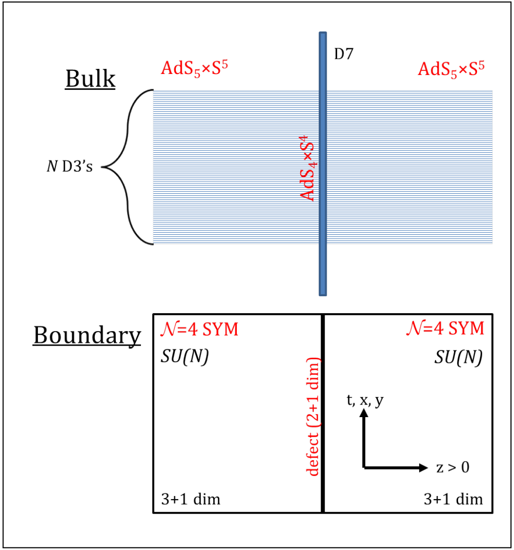
$$\mathfrak{n}(L, N_1, N_2, N_3) = \det G .$$

Section 2

One-point Functions in the D3-D7 System

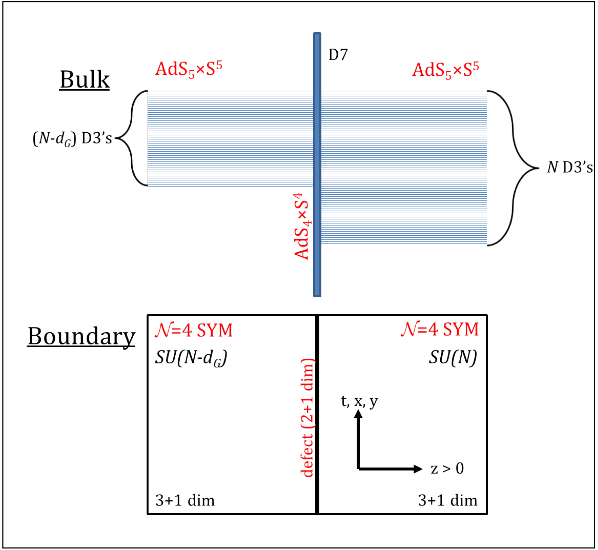
M. de Leeuw, C. Kristjansen, G. Linardopoulos, *One-point functions of non-protected operators in the $SO(5)$ symmetric D3-D7 dCFT*. J.Phys. A:Math.Theor., **50** (2017) 254001, [arXiv:1612.06236]

The $SO(5)$ symmetric D3-D7 system: description



- In the bulk, the D3-D7 system describes IIB superstring theory on $\text{AdS}_5 \times \text{S}^5$ bisected by a D7-brane with worldvolume geometry $\text{AdS}_4 \times \text{S}^4$.
- The dual field theory is still $SU(N)$, $\mathcal{N} = 4$ SYM in $3 + 1$ dimensions, that interacts with a CFT living on the $2 + 1$ dimensional defect:
$$S = S_{\mathcal{N}=4} + S_{2+1}.$$
- Due to the presence of the defect, the total bosonic symmetry of the system is reduced from $SO(4, 2) \times SO(6)$ to $SO(3, 2) \times SO(5)$.
- The relative co-dimension of the branes is $\#\text{ND} = 6 \rightarrow$ no unbroken supersymmetry.
- Tachyonic instability...

The $(\text{D3-D7})_{d_G}$ system

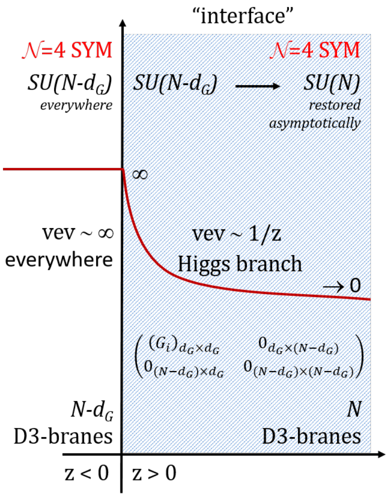


- To stabilize the system, add an instanton bundle on the S^4 component of the $AdS_4 \times S^4$ D7-brane, with instanton number $d_G = (n+1)(n+2)(n+3)/6$. (Myers-Wapler, 2008)
- Then exactly d_G of the N D3-branes ($N \gg d_G$) will end on the D7-brane.
- On the dual gauge theory side, the gauge group $SU(N) \times SU(N)$ breaks to $SU(N) \times SU(N - d_G)$.
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs... (Karch-Randall, 2001b)

Subsection 2

Nested one-point functions at tree-level

The dCFT interface of D3-D7



- As before, we need an interface to separate the $SU\,(N)$ and $SU\,(N-d_G)$ regions of the $(\text{D3-D7})_{d_G}$ dCFT...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N}=4$ SYM:

$$A_\mu = \psi_a = 0, \qquad \frac{d^2\Phi_i}{dz^2} = [\Phi_j, [\Phi_j, \Phi_i]]\,, \quad i,j = 1,\ldots,6.$$

- A manifestly $SO(5) \subset SO(3,2) \times SO(5)$ symmetric solution is given by $(z > 0)$:

$$\Phi_i\,(z) = \frac{G_i \oplus 0_{(N-d_G)\times(N-d_G)}}{\sqrt{8}\,z}, \quad i = 1,\ldots,5, \qquad \Phi_6 = 0\,.$$

Kristjansen-Semenoff-Young, 2012

The matrices G_i are known as "fuzzy" S^4 matrices or "G-matrices".

The "fuzzy" S^4 G-matrices

Here's the definition of the five $d_G \times d_G$ "fuzzy" S^4 matrices (G-matrices) G_i:

$$G_i \equiv \Bigg[\underbrace{\overbrace{\gamma_i \otimes \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4}^{n \text{ factors}} + \mathbb{1}_4 \otimes \gamma_i \otimes \ldots \otimes \mathbb{1}_4 + \ldots + \mathbb{1}_4 \otimes \ldots \otimes \mathbb{1}_4 \otimes \gamma_i}_{n \text{ terms}}\Bigg]_{\text{sym}} \qquad (i = 1, \ldots, 5),$$

Castelino-Lee-Taylor, 1997

where γ_i are the five 4×4 Euclidean Dirac matrices:

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

and σ_i are the three 2×2 Pauli matrices. The ten commutators of the five G-matrices,

$$G_{ij} \equiv \frac{1}{2}\left[G_i, G_j\right]$$

furnish a d_G-dimensional (anti-hermitian) irreducible representation of $\mathfrak{so}(5) \simeq \mathfrak{sp}(4)$:

$$\left[G_{ij}, G_{kl}\right] = 2\left(\delta_{jk} G_{il} + \delta_{il} G_{jk} - \delta_{ik} G_{jl} - \delta_{jl} G_{ik}\right).$$

The "fuzzy" S^4 G-matrices

The dimension of the G-matrices is equal to the instanton number $d_G = (n+1)(n+2)(n+3)/6$:

n	1	2	3	4	5	6	7	8	9	10	...
d_G	4	10	20	35	56	84	120	165	220	286	...

E.g., for $n = 2$, here are the 10×10 G-matrices:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

1-point functions

The 1-point functions of local gauge-invariant scalar operators

$$\langle \mathcal{O}(z, \mathbf{x}) \rangle = \frac{C}{z^{\Delta}}, \qquad z > 0,$$

can again be calculated within the D3-D7 dCFT from the corresponding fuzzy-funnel solution, e.g.

$$\boxed{\mathcal{O}(z, \mathbf{x}) = \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[\Phi_{i_1} \ldots \Phi_{i_L}\right] \xrightarrow[\text{interface}]{SO(5)} \frac{1}{8^{L/2} z^L} \cdot \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[G_{i_1} \ldots G_{i_L}\right]}$$

where $\Psi^{i_1 \ldots i_L}$ is an $\mathfrak{so}(6)$-symmetric tensor and the constant C is given by (MPS=*matrix product state*)

$$C = \frac{1}{\sqrt{L}} \left(\frac{\pi^2}{\lambda}\right)^{L/2} \cdot \frac{\langle \mathrm{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{\frac{1}{2}}}, \qquad \left\{ \begin{array}{ll} \langle \mathrm{MPS} | \Psi \rangle \equiv \Psi^{i_1 \ldots i_L} \mathrm{Tr}\left[G_{i_1} \ldots G_{i_L}\right] & (\text{"overlap"}) \\ \langle \Psi | \Psi \rangle \equiv \Psi^{i_1 \ldots i_L} \Psi_{i_1 \ldots i_L} & \end{array} \right\}.$$

- The mixing of single-trace operators up to one-loop in $\mathcal{N} = 4$ SYM is described by the integrable $\mathfrak{so}(6)$ spin chain of Minahan-Zarembo.
- We will assume that the above result is unaffected in the dCFT that is dual to the D3-D7 system.

Example: chiral primary operators

The one-point function of the chiral primary operators

$$\mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}}\left(\frac{8\pi^2}{\lambda}\right)^{L/2} \cdot C^{i_1\ldots i_L}\operatorname{Tr}\left[\Phi_{i_1}\ldots\Phi_{i_L}\right],$$

where $C^{i_1\ldots i_L}$ are symmetric & traceless tensors satisfying

$$C^{i_1\ldots i_L} C^{i_1\ldots i_L} = 1 \qquad \& \qquad Y_L = C^{i_1\ldots i_L}\hat{x}_{i_1}\ldots\hat{x}_{i_L}, \qquad \sum_{i=4}^{9}\hat{x}_i^2 = 1,$$

and $Y_L(\theta)$ is the $SO(5)$ spherical harmonic ($Y_{\text{odd}}(0) = 0$), have been calculated at weak coupling:

$$\langle\mathcal{O}_{\text{CPO}}(x)\rangle = \frac{d_G}{\sqrt{L}}\left(\frac{\pi^2 c_{\mathrm{G}}}{\lambda}\right)^{L/2}\frac{Y_L(0)}{z^L}, \qquad c_{\mathrm{G}} \equiv n(n+4), \qquad d_G \ll N \to \infty.$$

Kristjansen-Semenoff-Young, 2012

The large-n limit reproduces the supergravity calculation:

$$\langle\mathcal{O}_{\text{CPO}}(x)\rangle \xrightarrow{n\to\infty} \frac{Y_L(0)}{\sqrt{L}}\left(\frac{\pi^2 n^2}{\lambda}\right)^{L/2}\frac{n^3}{z^L}.$$

Bethe state overlaps

- The matrix product state projects the 3 complex scalars on the $SO(5)$ fuzzy funnel solution:

$$\langle \text{MPS}|\Psi\rangle = z^L \cdot \sum_{1\leq x_k\leq L} \psi(x_k) \cdot \text{Tr}\left[\mathcal{Z}^{x_1-1}\mathcal{W}\mathcal{Z}^{x_2-x_1-1}\mathcal{Y}\mathcal{Z}^{x_3-x_2-1}\overline{\mathcal{W}}\mathcal{Z}^{x_4-x_3-1}\overline{\mathcal{Y}}\ldots\right]$$

where the complex scalar fields $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$ are expressed in terms of the G-matrices as follows:

$$\mathcal{W} \sim G_1 + iG_2 \qquad \mathcal{Y} \sim G_3 + iG_4 \qquad \mathcal{Z} \sim G_5$$

$$\overline{\mathcal{W}} \sim G_1 - iG_2 \qquad \overline{\mathcal{Y}} \sim G_3 - iG_4 \qquad \overline{\mathcal{Z}} \sim G_5$$

- The corresponding matrix product state (MPS) is given by:

$$|\text{MPS}\rangle = \text{Tr}_a\left[\prod_{l=1}^{L}\left(|\mathcal{Z}\rangle_l \otimes G_5\right) + |\mathcal{W}\rangle_l \otimes (G_1 + iG_2) + |\mathcal{Y}\rangle_l \otimes (G_3 + iG_4) + \text{c.c.}\right].$$

It can be proven that all possible assignments for the fields $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$ are equivalent.

Subsection 3

Determinant formulas

$SO(5)$ vacuum overlap

For the vacuum overlap we have found:

$$\langle \mathrm{MPS}|0\rangle = \mathrm{Tr}\left[G_5^L\right] = \sum_{j=1}^{n+1}\left[j\,(n-j+2)\,(n-2j+2)^L\right].$$

Changing variables $j \leftrightarrow (n+2-j)$, an overall factor $(-1)^L$ comes out, leading the vacuum overlap to zero for L odd. Equivalently, we may write

$$\langle \mathrm{MPS}|0\rangle = 2^L\left[\frac{(n+2)^2}{4}\left(\zeta\left(-L,-\frac{n}{2}\right)-\zeta\left(-L,\frac{n}{2}+1\right)\right)-\left(\zeta\left(-L-2,-\frac{n}{2}\right)-\zeta\left(-L-2,\frac{n}{2}+1\right)\right)\right],$$

where the Hurwitz zeta function is defined as:

$$\zeta\,(s,\mathrm{a}) \equiv \sum_{n=0}^{\infty}\frac{1}{(n+\mathrm{a})^s}.$$

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$$\langle \text{MPS}|0\rangle = \begin{cases} 0, & L \text{ odd} \\ 2^L \cdot \left[\frac{2}{L+3}\, B_{L+3}\left(-\frac{n}{2}\right) - \frac{(n+2)^2}{2(L+1)}\, B_{L+1}\left(-\frac{n}{2}\right)\right], & L \text{ even}, \end{cases}$$

by using the relationship between the Hurwitz zeta function and the Bernoulli polynomials $B_m(x)$.

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by using the relationship between the Hurwitz zeta function and the Bernoulli polynomials $B_m(x)$.

In the large-n limit we find:

$$\langle \mathrm{MPS}|0\rangle \sim \frac{n^{L+3}}{2\,(L+1)\,(L+3)} + O\left(n^{L+2}\right), \qquad n \to \infty.$$

Overlap properties

- The overlaps $\langle \text{MPS}|\Psi\rangle$ of all the highest-weight eigenstates vanish unless:

$$\#\mathcal{W} = \#\overline{\mathcal{W}}, \qquad \#\mathcal{Y} = \#\overline{\mathcal{Y}}.$$

Therefore the only $\mathfrak{so}(6)$ eigenstates that have nonzero one-point functions are those with:

$$N_1 = 2N_2 = 2N_3 \equiv M \ \ (\text{even}).$$

Evidently, all one-point functions vanish in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors.

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Evidently, all one-point functions vanish in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors.

- Because the third conserved charge Q_3 annihilates the matrix product state:

$$Q_3 \cdot |\text{MPS}\rangle = 0,$$

all the one-point functions will vanish, unless all the Bethe roots are fully balanced:

$$\left\{u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0\right\}$$
$$\left\{v_1, \ldots, v_{N_+/2}, -v_1, \ldots, -v_{N_+/2}, 0\right\}, \qquad \left\{w_1, \ldots, w_{N_-/2}, -w_1, \ldots, -w_{N_-/2}, 0\right\}.$$

Example: the Konishi operator

- A prime example of a non-protected operator is the Konishi operator:

$$\mathcal{K} = \mathrm{Tr}\left[\Phi_{\mathrm{i}}\Phi_{\mathrm{i}}\right] = \mathrm{Tr}\left[\mathcal{Z}\overline{\mathcal{Z}}\right] + \mathrm{Tr}\left[\mathcal{W}\overline{\mathcal{W}}\right] + \mathrm{Tr}\left[\mathcal{Y}\overline{\mathcal{Y}}\right]$$

which is an eigenstate of the $\mathfrak{so}(6)$ Hamiltonian with $L = N_1 = 2$, $N_2 = N_3 = 1$ and eigenvalue:

$$E = 2 + \frac{3\lambda}{4\pi^2} + \dots$$

- Using the Casimir relation:

$$\mathrm{Tr}\left[G_i G_i\right] = \frac{1}{6} n\,(n+1)\,(n+2)\,(n+3)\,(n+4)$$

we can compute the one-point function of the Konishi operator:

$$\langle \mathcal{K} \rangle = \frac{1}{6\sqrt{3}} \frac{\pi^2}{\lambda} n\,(n+1)\,(n+2)\,(n+3)\,(n+4)\,.$$

The $L211$ states

- More generally, we can consider eigenstates with $N_1 = 2$, $N_2 = N_3 = 1$ and arbitrary L:

$$|p\rangle = \sum_{x_1<x_2} \left(e^{ip(x_1-x_2)} + e^{ip(x_2-x_1+1)}\right) \cdot |\ldots \underset{x_1}{\mathcal{X}} \ldots \underset{x_2}{\overline{\mathcal{X}}} \ldots\rangle - 2\sum_{x_3} \left(1 + e^{ip}\right) \cdot |\ldots \underset{x_3}{\overline{\mathcal{Z}}} \ldots\rangle,$$

where the dots stand for $\mathcal{Z}$, and $\mathcal{X}$ is any of the complex scalars $\mathcal{W}$, $\overline{\mathcal{W}}$, $\mathcal{Y}$, $\overline{\mathcal{Y}}$.

- The momentum p is found by solving the corresponding Bethe equations:

$$e^{ip(L+1)} = 1 \;\Rightarrow\; p = \frac{4m\pi}{L+1}, \qquad m = 1, \ldots, L+1$$

- Here's the one-loop energy of the $L211$ eigenstates:

$$E = L + \frac{\lambda}{\pi^2}\sin^2\left[\frac{2m\pi}{L+1}\right] + \ldots, \qquad m = 1, \ldots, L+1$$

The $L211$ determinant formula

- The corresponding one-point function for all n is given in terms of the $n=1$ one:

$$\langle \mathcal{O}_{L211} \rangle = \left[\frac{u^2}{u^2 - 1/2} \sum_{n \bmod 2}^{n} j^L \cdot \frac{(n+2)^2 - j^2}{8} \cdot \frac{[u^2 + \frac{(n+2)j+1}{4}][u^2 - \frac{(n+2)j-1}{4}]}{[u^2 + (\frac{j+1}{2})^2][u^2 + (\frac{j-1}{2})^2]} \right] \cdot \langle \mathcal{O}^{n=1}_{L211} \rangle$$

where

$$\langle \mathcal{O}^{n=1}_{L211} \rangle = 8 \sqrt{\frac{L}{L+1} \frac{u^2 - \frac{1}{2}}{u^2 + \frac{1}{4}}} \sqrt{\frac{u^2 + \frac{1}{4}}{u^2}}, \qquad u \equiv \frac{1}{2} \cot \frac{p}{2}.$$

- The results fully reproduce the numerical values (given in units of $(\pi^2/\lambda)^{L/2}/\sqrt{L}$):

L	$N_{1/2/3}$	eigenvalue γ	n=1	n=2	n=3	n=4
2	2 1 1	6	$20\sqrt{\frac{2}{3}}$	$40\sqrt{6}$	$140\sqrt{6}$	$1120\sqrt{\frac{2}{3}}$
4	2 1 1	$5+\sqrt{5}$	$20+\frac{44}{\sqrt{5}}$	$\frac{96}{5}\left(15+\sqrt{5}\right)$	$84\left(21-\sqrt{5}\right)$	$\frac{3584}{5}\left(10-\sqrt{5}\right)$
4	2 1 1	$5-\sqrt{5}$	$20-\frac{44}{\sqrt{5}}$	$288-\frac{96}{\sqrt{5}}$	$84\left(21+\sqrt{5}\right)$	$\frac{3584}{5}\left(10+\sqrt{5}\right)$
6	2 1 1	1.50604	3.57792	324.178	11338.3	98726
6	2 1 1	4.89008	9.90466	1724.55	19513.8	120347
6	2 1 1	7.60388	61.6252	1044.86	8830.95	49114.4

The $L422$ determinant formula

- For the eigenstates with $N_1 = 4,\ N_2 = N_3 = 2$ and $L =$ even, we find (work in progress):

$$\langle \mathcal{O}_{L422} \rangle = \sum_{n \bmod 2}^{n} j^L \cdot \frac{(n+2)^2 - j^2}{4} \cdot \frac{Q_1\left[\frac{i\sqrt{(n+2)j+1}}{2}\right]}{Q_1\left[\frac{i(j-1)}{2}\right] Q_1\left[\frac{i(j+1)}{2}\right]} \left[\left(1+(-1)^L\right) Q_1\left[\frac{\sqrt{(n+2)j-1}}{2}\right] - (5n-2)\left(Q_2[0] + (-1)^L\, Q_3[0]\right)\right] \sqrt{\frac{G^+}{G_-}}$$

- The results fully reproduce the corresponding numerical values for $n = 1$ and $n = 2$:

L	$N_{1/2/3}$	eigenvalue γ	n=1	n=2	n=3	n=4
4	4 2 2	$\frac{1}{2}\left(13+\sqrt{41}\right)$	$2\sqrt{1410+\frac{25970}{3\sqrt{41}}}$	$16\sqrt{3090+\frac{10710}{\sqrt{41}}}$	$14\sqrt{161490+\frac{140310}{\sqrt{41}}}$	$896\sqrt{690-\frac{670}{3\sqrt{41}}}$
4	4 2 2	$\frac{1}{2}\left(13-\sqrt{41}\right)$	$2\sqrt{1410-\frac{25970}{3\sqrt{41}}}$	$16\sqrt{3090-\frac{10710}{\sqrt{41}}}$	$14\sqrt{161490-\frac{140310}{\sqrt{41}}}$	$896\sqrt{690+\frac{670}{3\sqrt{41}}}$
6	4 2 2	8	4.76832	2899.14	37483.7	247800
6	4 2 2	2.26228	8.68876	1090.46	11963	166654
6	4 2 2	3.81374	13.8862	4479.21	43679.9	238186
6	4 2 2	5.33676	22.5105	2995.7	34577.8	216443
6	4 2 2	8.94875	78.0614	1813.66	16647.9	95264.6
6	4 2 2	10.1954	138.297	151.877	10250	80604.6
6	4 2 2	12.4431	369.992	4881.61	33331.2	159221

The $L422$ formula reduces to the previous one for $N_1 = 2,\ N_2 = N_3 = 1$.

Section 3

Summary

Summary

We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric $(\text{D3-D5})_k$ dCFT and the $SO(5)$ symmetric $(\text{D3-D7})_{d_G}$ dCFT...

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We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric $(\text{D3-D5})_k$ dCFT and the $SO(5)$ symmetric $(\text{D3-D7})_{d_G}$ dCFT...

D3-D5 dCFT

- Because $Q_3 \cdot |\text{MPS}\rangle = 0$, all 1-point functions vanish unless the Bethe roots are fully balanced:

$$\{u_{1,i}\} = \{-u_{1,i}\}, \qquad \{u_{2,i}\} = \{-u_{2,i}\}, \qquad \{u_{3,i}\} = \{-u_{3,i}\}.$$

- In $\mathfrak{su}(2)$, all 1-point functions (vacuum included) vanish if M or L is odd.
- In $\mathfrak{su}(3)$, all 1-point functions vanish if (1) M is odd or (2) $L + N_+$ is odd.
- In $\mathfrak{so}(6)$, all 1-point functions vanish if (1) M is odd or (2) $L + N_+ + N_-$ is odd.
- We have found a determinant formula for the eigenstates, valid for all values of the flux k:

$$C_k(\{u_j; v_j; w_j\}) = \mathbb{T}_{k-1}(0) \cdot \sqrt{\frac{Q_1(0)\, Q_1(i/2)\, Q_1(ik/2)\, Q_1(ik/2)}{R_2(0)\, R_2(i/2)\, R_3(0)\, R_3(i/2)} \cdot \frac{\det G^+}{\det G^-}}$$

Summary

We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric $(\text{D3-D5})_k$ dCFT and the $SO(5)$ symmetric $(\text{D3-D7})_{d_G}$ dCFT...

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$$\{u_{1,i}\} = \{-u_{1,i}\}, \qquad \{u_{2,i}\} = \{-u_{2,i}\}, \qquad \{u_{3,i}\} = \{-u_{3,i}\}.$$

- Besides the vacuum, all 1-pt functions vanish in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors.
- In $\mathfrak{so}(6)$ all 1-point functions vanish unless $N_1 = 2N_2 = 2N_3 \equiv M$ (even).
- The vacuum also vanishes when $L = \text{odd}$.
- We have found a determinant formula for $L211$ eigenstates, valid for all values of the instanton number n:

$$\langle \mathcal{O}_{L211} \rangle = \left[\frac{u^2}{u^2 - 1/2} \sum_{n \bmod 2}^{n} j^L \cdot \frac{(n+2)^2 - j^2}{8} \cdot \frac{[u^2 + \frac{(n+2)j+1}{4}][u^2 - \frac{(n+2)j-1}{4}]}{[u^2 + (\frac{j+1}{2})^2][u^2 + (\frac{j-1}{2})^2]}\right] \cdot \langle \mathcal{O}_{L211}^{n=1} \rangle$$

Thank you!