Nested One-point Functions in AdS/dCFT

Georgios Linardopoulos
NCSR "Demokritos" and National & Kapodistrian University of Athens

Workshop on higher-point correlation functions and integrable AdS/CFT
Hamilton Mathematics Institute – Trinity College Dublin, April 16th 2018

One-point functions in the D3-D5 system

One-point functions in the D3-D7 system

Summary

Table of Contents

1. One-point Functions in the D3-D5 System
   - Introducing the D3-D5 system
   - Nested one-point functions at tree-level
   - $su(2)_k$ representations
   - Determinant formulas

2. One-point Functions in the D3-D7 System
   - Introducing the D3-D7 system
   - Nested one-point functions at tree-level
   - Determinant formulas

3. Summary
Section 1

One-point Functions in the D3-D5 System
The D3-D5 system: description

- In the bulk, the D3-D5 system describes IIB Superstring theory on $\text{AdS}_5 \times S^5$ bisected by D5 branes with worldvolume geometry $\text{AdS}_4 \times S^2$.

- The dual field theory is still $SU(N)$, $\mathcal{N} = 4$ SYM in $3 + 1$ dimensions, that now interacts with a SCFT that lives on the $2+1$ dimensional defect.

- Due to the presence of the defect, the total bosonic symmetry of the system is reduced from $SO(4, 2) \times SO(6)$ to $SO(3, 2) \times SO(3) \times SO(3)$.

- The corresponding superalgebra $\mathfrak{psu}(2, 2|4)$ becomes $\mathfrak{osp}(4|4)$.
The $(D3-D5)_k$ system

- Add $k$ units of background $U(1)$ flux on the $S^2$ component of the $AdS_4 \times S^2$ D5-brane.
- Then $k$ of the $N$ D3-branes ($N \gg k$) will end on the D5-brane.
- On the dual SCFT side, the gauge group $SU(N) \times SU(N)$ breaks to $SU(N-k) \times SU(N)$.
- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs...

(Karch-Randall, 2001b)
Subsection 2

Nested one-point functions at tree-level
The dCFT interface of D3-D5

- An interface is a wall between two (different/same) QFTs
- It can be described by means of classical solutions that are known as "fuzzy-funnel" solutions (Constable-Myers-Tafjord, 1999 & 2001)
- Here, we need an interface to separate the SU($N$) and SU($N-k$) regions of the (D3-D5)$_k$ dCFT...
- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:

  $$A_\mu = \psi_a = 0, \quad \frac{d^2 \Phi_i}{dz^2} = [\Phi_j, [\Phi_j, \Phi_i]], \quad i, j = 1, \ldots, 6.$$  

- A manifestly $SO(3) \simeq SU(2)$ symmetric solution is given by ($z > 0$):

  $$\Phi_{2i-1}(z) = \frac{1}{z} \begin{pmatrix} (t_i)_{k \times (N-k)} & 0_{k \times (N-k)} \\ 0_{(N-k) \times k} & 0_{(N-k) \times (N-k)} \end{pmatrix} \quad & \Phi_{2i} = 0,$

  Nagasaki-Yamaguchi, 2012

  where the matrices $t_i$ furnish a k-dimensional representation of $su(2)$:

  $$[t_i, t_j] = i \epsilon_{ijk} t_k.$$
We use the following \( k \times k \) dimensional representation of \( \mathfrak{su}(2) \):

\[
\begin{align*}
t_+ &= \sum_{i=1}^{k-1} c_{k,i} E_{i+1}^i, \\
t_- &= \sum_{i=1}^{k-1} c_{k,i} E_{i+1}^i, \\
t_3 &= \sum_{i=1}^{k} d_{k,i} E_i^i \\
t_1 &= \frac{t_+ + t_-}{2}, \\
t_2 &= \frac{t_+ - t_-}{2i}
\end{align*}
\]

\[
\begin{align*}
c_{k,i} &= \sqrt{i(k-i)}, \\
d_{k,i} &= \frac{1}{2}(k-2i+1),
\end{align*}
\]

where \( E_{ij} \) are the standard matrix unities that are zero everywhere except \( (i,j) \) where they’re 1.
1-point functions

Following Nagasaki & Yamaguchi (2012), the 1-point functions of local gauge-invariant scalar operators

$\langle \mathcal{O}(z, x) \rangle = \frac{C}{z^\Delta}, \quad z > 0,$

can be calculated within the D3-D5 dCFT from the corresponding fuzzy-funnel solution, for example:

$\mathcal{O}(z, x) = \Psi_{i_1 \ldots i_L} \text{Tr} \left[ \Phi_{2i_1 - 1} \ldots \Phi_{2i_L - 1} \right] \xrightarrow{\text{interface}} \frac{1}{z^L} \cdot \Psi_{i_1 \ldots i_L} \text{Tr} \left[ t_{i_1} \ldots t_{i_L} \right]$

where $\Psi_{i_1 \ldots i_L}$ is an $\mathfrak{so}(6)$-symmetric tensor and the constant $C$ is given by (MPS = matrix product state)

$C = \frac{1}{\sqrt{L}} \left( \frac{8\pi^2}{\lambda} \right)^{L/2} \cdot \frac{\langle \text{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}} = \left\{ \begin{array}{l}
\langle \text{MPS} | \Psi \rangle \equiv \Psi_{i_1 \ldots i_L} \text{Tr} \left[ t_{i_1} \ldots t_{i_L} \right] \quad ("\text{overlap")}
\langle \Psi | \Psi \rangle \equiv \Psi_{i_1 \ldots i_L} \Psi_{i_1 \ldots i_L}
\end{array} \right\},$

which ensures that the 2-point function will be normalized to unity ($\mathcal{O} \rightarrow (2\pi)^L \cdot \mathcal{O}/(\lambda^{L/2}\sqrt{L})$

$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}$

within $SU(N), \mathcal{N} = 4$ SYM (i.e. without the defect).
Example: chiral primary operators

The one-point functions of the chiral primary operators

\[ \mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}} \left( \frac{8\pi^2}{\lambda} \right)^{L/2} \cdot C^{i_1 \ldots i_L} \text{Tr}[\Phi_{i_1} \ldots \Phi_{i_L}] , \]

where \( C^{i_1 \ldots i_L} \) are symmetric & traceless tensors satisfying

\[ C^{i_1 \ldots i_L} C^{i_1 \ldots i_L} = 1 \quad \& \quad Y_L = C^{i_1 \ldots i_L} \hat{x}_{i_1} \ldots \hat{x}_{i_L} , \quad \sum_{i=4}^{6} \hat{x}_i^2 = \cos^2 \psi , \quad \sum_{i=7}^{9} \hat{x}_i^2 = \sin^2 \psi \]

and \( Y_L(\psi) \) is the \( SO(3) \times SO(3) \subseteq SO(6) \) spherical harmonic, have been calculated at weak coupling:

\[ \langle \mathcal{O}_{\text{CPO}}(x) \rangle = \frac{1}{\sqrt{L}} \left( \frac{2\pi^2}{\lambda} \right)^{L/2} k \left( k^2 - 1 \right)^{L/2} \frac{Y_L(\pi/2)}{z^L} , \quad k \ll N \to \infty . \]

\[ \text{Nagasaki-Yamaguchi, 2012} \]

The large-\( k \) limit agrees with the supergravity calculation at tree-level:

\[ \langle \mathcal{O}_{\text{CPO}}(x) \rangle = \frac{k^{L+1}}{\sqrt{L}} \left( \frac{2\pi^2}{\lambda} \right)^{L/2} \frac{Y_L(\pi/2)}{z^L} \cdot \left[ 1 + \frac{\lambda I_1}{\pi^2 k^2} + \ldots \right] , \quad I_1 \equiv \frac{3}{2} + \frac{(L-2)(L-3)}{4(L-1)} . \]
Dilatation operator

The mixing of single-trace operators $O(x)$ is generally described by the integrable $so(6)$ spin chain:

$$
\mathcal{D} = L \cdot I + \frac{\lambda}{8\pi^2} \cdot H + \sum_{n=2}^{\infty} \lambda^n \cdot D_n, \quad H = \sum_{j=1}^{L} \left( I_{j,j+1} - P_{j,j+1} + \frac{1}{2} K_{j,j+1} \right), \quad \lambda = g_{YM}^2 N,
$$

up to one loop in $\mathcal{N} = 4$ SYM, where

$$
I \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = |\ldots \Phi_a \Phi_b \ldots\rangle
$$

$$
P \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = |\ldots \Phi_b \Phi_a \ldots\rangle
$$

$$
K \cdot |\ldots \Phi_a \Phi_b \ldots\rangle = \delta_{ab} \sum_{c=1}^{6} |\ldots \Phi_c \Phi_c \ldots\rangle.
$$

The above result is unaffected by the presence of a defect in the SCFT (DeWolfe-Mann, 2004).
Bethe eigenstates

In the following we will examine eigenstates of the $\mathfrak{so}(6)$ spin chain which can be written as:

$$|\Psi\rangle \equiv \sum_{x_i} \psi_i (u_1, u_2, u_3) \cdot |\bullet \ldots \bullet \uparrow \bullet \ldots \bullet \downarrow \bullet \ldots \bullet \downarrow \ldots \rangle,$$

where $u_{1,2,3}$ are the rapidities of the excitations at $x_i$. The corresponding single-trace operator is

$$|\bullet \ldots \bullet \uparrow \bullet \ldots \bullet \downarrow \bullet \ldots \bullet \downarrow \ldots \rangle \sim \text{Tr} \left[ Z^{x_1-1} W Z^{x_2-x_1-1} Y Z^{x_3-x_2-1} \bar{W} Z^{x_4-x_3-1} \bar{Y} \ldots \right],$$

where $Z$ (ground state field), $W$, $Y$ (excitations) are the following three complex scalars:

$$W = \Phi_1 + i\Phi_2 \sim \uparrow \quad Y = \Phi_3 + i\Phi_4 \sim \downarrow \quad Z = \Phi_5 + i\Phi_6 \sim \bullet$$

$$\bar{W} = \Phi_1 - i\Phi_2 \sim \uparrow \quad \bar{Y} = \Phi_3 - i\Phi_4 \sim \downarrow \quad \bar{Z} = \Phi_5 - i\Phi_6 \sim \circ$$

The wavefunction $\psi (u_1, u_2, u_3)$ can be constructed with the (nested) coordinate Bethe ansatz...
Let us first construct the kets $\left| {\ldots \uparrow x_1 \ldots \downarrow x_2 \ldots \uparrow x_3 \ldots \downarrow x_4 \ldots} \right>$.
Let us first construct the kets $|\cdots \uparrow \cdots \downarrow \cdots \uparrow \cdots \downarrow \cdots\rangle$.

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".
Let us first construct the kets $|\cdots\uparrow\cdots\downarrow\cdots\uparrow\cdots\downarrow\cdots\rangle$.

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

Start from a closed $so(6)$ spin chain of length $L$: 

```

```
Let us first construct the kets $\langle \bullet \ldots \bullet \uparrow \ldots \bullet \downarrow \ldots \bullet \uparrow \ldots \bullet \downarrow \ldots \rangle$. 

Because the excitations can have 5 different polarizations, we apply a procedure called ”nesting”.

Start from a closed $\mathfrak{so}(6)$ spin chain of length $L$. Excite exactly $N_1$ sites of the chain:
Let us first construct the kets $|\cdots \uparrow \cdots \downarrow \cdots \uparrow \cdots \downarrow \cdots \rangle_{x_1 \ x_2 \ x_3 \ x_4}$.

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

Start from a closed $so(6)$ spin chain of length $L$. Excite exactly $N_1$ sites of the chain:

Now take the $N_1$ excitations to be the ground state.
Let us first construct the kets $|\cdots \uparrow \cdots \downarrow \cdots \uparrow \cdots \rangle$. Because the excitations can have 5 different polarizations, we apply a procedure called ”nesting”.

Start from a closed $so(6)$ spin chain of length $L$. Excite exactly $N_1$ sites of the chain:

Now take the $N_1$ excitations to be the ground state. Excite $N_2$ sites of the new chain...
Let us first construct the kets $|\cdots \uparrow \cdots \downarrow \cdots \uparrow \cdots \downarrow \cdots \rangle$.

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

Start from a closed $so(6)$ spin chain of length $L$. Excite exactly $N_1$ sites of the chain:

Now take the $N_1$ excitations to be the ground state. Excite $N_2$ sites of the new chain... or $N_3$ sites:
Let us first construct the kets \( | \bullet \cdots \uparrow \bullet \cdots \downarrow \bullet \cdots \uparrow \bullet \cdots \rangle \).

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

Start from a closed \( so(6) \) spin chain of length \( L \). Excite exactly \( N_1 \) sites of the chain:

Now take the \( N_1 \) excitations to be the ground state. Excite \( N_2 \) sites of the new chain... or \( N_3 \) sites:

We end up with three sets/levels of rapidities, one rapidity for each excitation:

\[
\begin{align*}
  u_1 &= \{ u_{1,j} \}_{j=1}^{N_1}, \\
  u_2 &= \{ u_{2,j} \}_{j=1}^{N_2}, \\
  u_3 &= \{ u_{3,j} \}_{j=1}^{N_3},
\end{align*}
\]

each set corresponds to a simple root \( \alpha_{1,2,3} \) of \( so(6) \).
Let us first construct the kets \(|\cdots \uparrow \downarrow \uparrow \downarrow \cdots \rangle\).

Because the excitations can have 5 different polarizations, we apply a procedure called "nesting".

Start from a closed \(\mathfrak{so}(6)\) spin chain of length \(L\). Excite exactly \(N_1\) sites of the chain:

Now take the \(N_1\) excitations to be the ground state. Excite \(N_2\) sites of the new chain... or \(N_3\) sites:

We end up with three sets/levels of rapidities, one rapidity for each excitation:

\[
\mathbf{u}_1 = \{u_{1,j}\}_{j=1}^{N_1}, \quad \mathbf{u}_2 = \{u_{2,j}\}_{j=1}^{N_2}, \quad \mathbf{u}_3 = \{u_{3,j}\}_{j=1}^{N_3},
\]

each set corresponds to a simple root \(\alpha_{1,2,3}\) of \(\mathfrak{so}(6)\).

To construct the kets, we must map the sets of rapidities to the available complex scalar fields.
As we’ve just seen, each set of rapidities can be associated to a node of the so(6) Dynkin diagram:

\[ (0 \leq N_1 \leq L, \ 0 \leq N_2 \leq N_1/2, \ 0 \leq N_3 \leq N_2). \]

The total weight of the so(6) representation will then be given by:

\[ w = Lq - N_1 \alpha_1 - N_2 \alpha_2 - N_3 \alpha_3 \]

where \( q \equiv (1, 0, 0) \) and the so(6) roots are \( \alpha_1 \equiv (1, -1, 0), \alpha_2 \equiv (0, 1, -1), \alpha_3 \equiv (0, 1, 1). \)
Rapidities & fields

- As we've just seen, each set of rapidities can be associated to a node of the $\mathfrak{so}(6)$ Dynkin diagram:

\[ N_1 \quad N_2 \quad N_3 \]

\[(0 \leq N_1 \leq L, \ 0 \leq N_2 \leq N_1/2, \ 0 \leq N_3 \leq N_2) .\]

- The total weight of the $\mathfrak{so}(6)$ representation will then be given by:

\[ w = Lq - N_1 \alpha_1 - N_2 \alpha_2 - N_3 \alpha_3 \]

where \( q \equiv (1, 0, 0) \) and the $\mathfrak{so}(6)$ roots are \( \alpha_1 \equiv (1, -1, 0) \), \( \alpha_2 \equiv (0, 1, -1) \), \( \alpha_3 \equiv (0, 1, 1) \).

- The corresponding Cartan charges are given by:

\[ w = (J_1, J_2, J_3) = (L - N_1, N_1 - N_2 - N_3, N_2 - N_3), \quad J_1 \geq J_2 \geq J_3 \geq 0. \]
As we've just seen, each set of rapidities can be associated to a node of the \( \mathfrak{so} (6) \) Dynkin diagram:

\[
\begin{align*}
\bullet & \quad N_1 \\
\bullet & \quad N_2 \\
\bullet & \quad N_3
\end{align*}
\]

\((0 \leq N_1 \leq L, \ 0 \leq N_2 \leq N_1/2, \ 0 \leq N_3 \leq N_2)\).

The total weight of the \( \mathfrak{so} (6) \) representation will then be given by:

\[
w = Lq - N_1 \alpha_1 - N_2 \alpha_2 - N_3 \alpha_3
\]

where \( q \equiv (1, 0, 0) \) and the \( \mathfrak{so} (6) \) roots are \( \alpha_1 \equiv (1, -1, 0), \ \alpha_2 \equiv (0, 1, -1), \ \alpha_3 \equiv (0, 1, 1) \).

Here are the corresponding Dynkin indices:

\[
[w \cdot \alpha_2, w \cdot \alpha_1, w \cdot \alpha_3] = [J_2 - J_3, J_1 - J_2, J_2 + J_3] = [N_1 - 2N_2, L - 2N_1 + N_2 + N_3, N_1 - 2N_3].
\]
Rapidities & fields

- As we've just seen, each set of rapidities can be associated to a node of the \( \mathfrak{so}(6) \) Dynkin diagram:

\[
\begin{align*}
N_1 & \quad (0 \leq N_1 \leq L, \ 0 \leq N_2 \leq N_1/2, \ 0 \leq N_3 \leq N_2) .
\end{align*}
\]

- The total weight of the \( \mathfrak{so}(6) \) representation will then be given by:

\[
w = Lq - N_1 \alpha_1 - N_2 \alpha_2 - N_3 \alpha_3
\]

where \( q \equiv (1, 0, 0) \) and the \( \mathfrak{so}(6) \) roots are \( \alpha_1 \equiv (1, -1, 0), \ \alpha_2 \equiv (0, 1, -1), \ \alpha_3 \equiv (0, 1, 1) \).

- Each complex scalar field is associated to the following set of weights:

\[
\begin{align*}
\mathcal{Z} & \sim q & \mathcal{W} & \sim q - \alpha_1 & \mathcal{Y} & \sim q - \alpha_1 - \alpha_2 \\
\overline{\mathcal{Z}} & \sim q - 2\alpha_1 - \alpha_2 - \alpha_3 & \overline{\mathcal{W}} & \sim q - \alpha_1 - \alpha_2 - \alpha_3 & \overline{\mathcal{Y}} & \sim q - \alpha_1 - \alpha_3
\end{align*}
\]
Nested Bethe Ansatz

Here’s the nested \( so(6) \) wavefunction (in a somewhat simplified form):

\[
\psi_i (u_1, u_2, u_3) = \sum_{P_1} A_1 (P_1) \prod_{j=1}^{N_1} \frac{1}{u_{1, P_1, j} - i/2} \left( \frac{u_{1, P_1, j} + i/2}{u_{1, P_1, j} - i/2} \right)^{n_{1,j}^{-1}} \cdot \psi_{(2,i)} (u_1, u_2) \cdot \psi_{(3,i)} (u_1, u_3)
\]

where

\[
\psi_{(a,i)} (u_1, u_a) = \sum_{P_a} A_a (P_a) \prod_{j=1}^{N_a} \frac{1}{u_{a, P_a, j} - u_{1, P_1, n_{a,j}} - i/2} \prod_{k=1}^{n_{a,j}^{-1}} \frac{u_{a, P_a, j} - u_{1, P_1, k} + i/2}{u_{a, P_a, j} - u_{1, P_1, k} - i/2}, \quad a = 2, 3,
\]

and

\[
A_a (\ldots, k, j, \ldots) = A_a (\ldots, j, k, \ldots) S_a (u_{a, k}, u_{a, j}), \quad S_a (u_{a, k}, u_{a, j}) \equiv \frac{u_{a, k} - u_{a, j} + i}{u_{a, k} - u_{a, j} - i}.
\]
Bethe equations

- The periodicity of the Bethe wavefunction $\psi$ (at each nesting level) leads to the Bethe equations:

  $\left( \frac{u_{1,i} + i/2}{u_{1,i} - i/2} \right)^L = \prod_{j \neq i}^{N_1} u_{1,i} - u_{1,j} + i \prod_{k=1}^{N_2} u_{1,i} - u_{2,k} - i/2 \prod_{l=1}^{N_3} u_{1,i} - u_{3,l} - i/2$, \quad $i = 1, \ldots, N_1 \equiv M$

  $1 = \prod_{l \neq i}^{N_2} u_{2,i} - u_{2,l} + i \prod_{k=1}^{N_1} u_{2,i} - u_{1,k} - i/2 \prod_{l=1}^{N_3} u_{2,i} - u_{3,l} - i/2$, \quad $i = 1, \ldots, N_2 \equiv N_+$

  $1 = \prod_{l \neq i}^{N_3} u_{3,i} - u_{3,l} + i \prod_{k=1}^{N_1} u_{3,i} - u_{1,k} - i/2 \prod_{l=1}^{N_3} u_{3,i} - u_{3,l} - i/2$, \quad $i = 1, \ldots, N_3 \equiv N_-$

  which must be satisfied by the rapidities of the excitations/Bethe roots.

- Because of the cyclicity of the trace, the momentum carrying roots obey the following relation:

  $\prod_{i=1}^{N_1} \frac{u_{1,i} + i/2}{u_{1,i} - i/2} = 1 \iff \sum_{i=1}^{N_1} p_{1,i} = 0$ \quad (momentum conservation).
The matrix product state projects the 3 complex scalars on the SU(2) fuzzy funnel solution:

\[
\langle \text{MPS} | \Psi \rangle = z^L \cdot \sum_{1 \leq x_k \leq L} \psi(x_k) \cdot \text{Tr} \left[ Z^{x_1-1} W Z^{x_2-1} Y Z^{x_3-1} W Z^{x_4-1} Y \ldots \right]
\]

where the complex scalar fields $Z$, $W$, $Y$ are expressed in terms of the $su(2)$ matrices as follows:

\[
W = \overline{W} = \frac{t_1}{z}, \quad Y = \overline{Y} = \frac{t_2}{z}, \quad Z = \overline{Z} = \frac{t_3}{z}
\]

The corresponding matrix product state (MPS) is given by:

\[
| \text{MPS} \rangle = \text{Tr}_a \left[ \prod_{l=1}^{L} | Z \rangle \otimes t_3 + | W \rangle \otimes t_1 + | Y \rangle \otimes t_2 + \text{c.c.} \right].
\]
The $\mathfrak{su}(2)$ subsector

For example, let us first consider the subsector that contains only two complex scalars:

$$\mathcal{W} = \Phi_1 + i\Phi_2 \quad \leftrightarrow \quad \left| \uparrow \right> \sim t_1$$

$$\mathcal{Z} = \Phi_5 + i\Phi_6 \quad \leftrightarrow \quad \left| \bullet \right> \sim t_3.$$

This is also known as the $\mathfrak{su}(2)$ subsector of the dCFT. In the $\mathfrak{su}(2)$ subsector, the trace operator $K_{j,j+1}$ does not contribute to the mixing matrix $D$:

$$H_{\mathfrak{su}(2)} = \sum_{j=1}^{L} (\mathbb{I}_{j,j+1} - P_{j,j+1}).$$

This is just the Hamiltonian of the Heisenberg $XXX_{1/2}$ spin chain. The MPS can be written as follows:

$$\left| \text{MPS} \right> = \text{Tr}_a \left[ \prod_{j=1}^{L} \left( \left| \uparrow_j \right> \otimes t_1 + \left| \bullet_j \right> \otimes t_3 \right) \right],$$

and it corresponds to the above choice of fields.
In the $\mathfrak{su}(2)$ subsector, $|\Psi\rangle$ is just the coordinate Bethe state $|p\rangle$:

$$
|p\rangle = \mathcal{N} \cdot \sum_{\sigma \in S_M} \sum_{1 \leq n_1 \leq \ldots \leq n_M \leq L} \exp \left[ i \sum_k p_{\sigma(k)} n_k + \frac{i}{2} \sum_{j < k} \theta_{\sigma(j)\sigma(k)} \right] |x\rangle,
$$

where

$$
|x\rangle \equiv |x_1, x_2, \ldots, x_M\rangle \equiv |1\ldots1 \uparrow 1\ldots1 \uparrow 1\ldots1\uparrow \ldots \uparrow \rangle = S_{n_1}^{-} \cdots S_{n_M}^{-} |0\rangle
$$

and the vacuum state $|0\rangle$ and the raising and lowering operators $S^\pm$ have been defined as

$$
|0\rangle = \bigotimes_{i=1}^L |\cdot\rangle, \quad S^+ |\uparrow\rangle = |\cdot\rangle \quad \& \quad S^- |\cdot\rangle = |\uparrow\rangle.
$$

The matrix $\theta_{jk}$ and the normalization constant $\mathcal{N}$ are given by:

$$
e^{i\theta_{jk}} = \frac{u_j - u_k + i}{u_j - u_k - i} \equiv S_{jk}, \quad u_j = \frac{1}{2} \cot \frac{p_j}{2}, \quad \mathcal{N} \equiv \exp \left[ -\frac{i}{2} \sum_{j < k} \theta_{jk} \right].$$
The $\mathfrak{su}(3)$ and $\mathfrak{so}(6)$ subsectors

- In the $\mathfrak{su}(3)$ subsector all the three real complex scalars contribute:

$$W = \Phi_1 + i \Phi_2 \sim t_1, \quad Y = \Phi_3 + i \Phi_4 \sim t_2, \quad Z = \Phi_5 + i \Phi_6 \sim t_3.$$ 

The corresponding wavefunction is constructed by means of the nested coordinate Bethe ansatz:

$$\psi = \sum_{P_1, P_2} A_1(P_1) A_2(P_2) \prod_{j=1}^{N_1} \prod_{j=1}^{N_2} \left( \frac{u_{1,P_1,j} + i/2}{u_{1,P_1,j} - i/2} \right)^{n_{1,j}} \prod_{k=1}^{n_{2,j}} \frac{u_{2,P_2,j} - u_{1,P_1,k} + i/2}{u_{2,P_2,j} - u_{1,P_1,k} - i/2} \delta_{k \neq n_{2,j}}$$

$$A_a(\ldots, k, j, \ldots) = A_a(\ldots, j, k, \ldots) S_a(u_{a,k}, u_{a,j}), \quad S_a(u_{a,k}, u_{a,j}) \equiv \frac{u_{a,k} - u_{a,j} + i}{u_{a,k} - u_{a,j} - i}.$$ 

- In the $\mathfrak{so}(6)$ subsector all the three real complex scalars contribute:

$$W = \overline{W} = \Phi_1 + i \Phi_2 \sim t_1, \quad Y = \overline{Y} = \Phi_3 + i \Phi_4 \sim t_2, \quad Z = \overline{Z} = \Phi_5 + i \Phi_6 \sim t_3$$

and similarly the $\mathfrak{so}(6)$ wavefunction can be constructed by the nested Bethe ansatz.
Subsection 4

Determinant formulas

1-point functions in $\mathfrak{su}(2)$

In the $\mathfrak{su}(2)$ sector our goal is to calculate the one-point function coefficient:

$$C = \frac{1}{\sqrt{L}} \left( \frac{8\pi^2}{\lambda} \right)^{L/2} \cdot \frac{\langle \text{MPS}|p \rangle}{\langle p|p \rangle^{1/2}}, \quad k \ll N \to \infty.$$ 

where the $k \times k$ matrices $t_{1,3}$ form a $k$-dimensional representation of $\mathfrak{su}(2)$:

$$\langle \text{MPS}|p \rangle = \mathcal{N} \cdot \sum_{\sigma \in S_M} \sum_{1 \leq x_k \leq L} \exp \left[ i \sum_k p_{\sigma(k)} x_k + \frac{i}{2} \sum_{j < k} \theta_{\sigma(j)\sigma(k)} \right] \cdot \text{Tr} \left[ t_{3x_1-1} t_1 t_{3x_2-x_1-1}^2 \cdots \right].$$

Overlap properties:

- The overlap $\langle \text{MPS}|p \rangle$ vanishes if $M \equiv N_1$ or $L$ is odd: $\text{Tr} \left[ t_{3x_1-1} t_1 t_{3x_2-x_1-1}^2 \cdots \right] \bigg|_{M \text{ or } L \text{ odd}} = 0$
- The overlap $\langle \text{MPS}|p \rangle$ vanishes if $\sum p_i \neq 0$: due to trace cyclicity
- The overlap $\langle \text{MPS}|p \rangle$ vanishes if momenta are not fully balanced ($p_i, -p_i$): due to $Q_3 \cdot |\text{MPS} \rangle = 0$

\textit{de Leeuw-Kristjansen-Zarembo, 2015}
The $\mathfrak{su}(2)$ determinant formula

Vacuum overlap:

$$\langle \text{MPS}|0\rangle = \text{Tr} \left[ t_3^L \right] = \zeta \left( -L, \frac{1-k}{2} \right) - \zeta \left( -L, \frac{1+k}{2} \right), \quad \zeta (s, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

where $\zeta (s, a)$ is the Hurwitz zeta function. For $M$ balanced excitations the overlap becomes:

$$C_k (\{u_j\}) \equiv \frac{\langle \text{MPS}|\{u_j\}\rangle_k}{\sqrt{\langle \{u_j\}|\{u_j\}\rangle}} = C_2 (\{u_j\}) \cdot \sum_{j=(1-k)/2}^{(k-1)/2} j^L \prod_{l=1}^{M/2} \frac{u_l^2 (u_l^2 + k^2/4)}{u_l^2 + (j-1/2)^2 [u_l^2 + (j + 1/2)^2]}$$

where

$$C_2 (\{u_j\}) \equiv \frac{\langle \text{MPS}|\{u_j\}\rangle_{k=2}}{\sqrt{\langle \{u_j\}|\{u_j\}\rangle}} = \left[ \prod_{j=1}^{M/2} \frac{u_j^2 + 1/4 \ det \ G^+}{u_j^2 \ det \ G^-} \right]^{1/2},$$

and the $M/2 \times M/2$ matrices $G_{jk}^\pm$ and $K_{jk}^\pm$ are defined as:

$$G_{jk}^\pm = \left( \frac{L}{u_j^2 + 1/4} - \sum_n K_{jn}^\pm \right) \delta_{jk} + K_{jk}^\pm \quad \& \quad K_{jk}^\pm = \frac{2}{1 + (u_j - u_k)^2} \pm \frac{2}{1 + (u_j + u_k)^2}.$$
Moving to the $\mathfrak{su}(3)$ sector, let us define the following Baxter functions $Q$ and $R$:

$$
Q_1(x) = \prod_{i=1}^{M} (x - u_i), \quad Q_2(x) = \prod_{i=1}^{N_+} (x - v_i), \quad R_2(x) = \prod_{i=1}^{2[N_+/2]} (x - v_i).
$$

All the one-point functions in the $\mathfrak{su}(3)$ sector are then given by

$$
C_k (\{u_j; v_j\}) = T_{k-1} (0) \cdot \sqrt{\frac{Q_1(0) Q_1(i/2)}{R_2(0) R_2(i/2)}} \cdot \frac{\det G^+}{\det G^-}
$$

where $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$ and

$$
T_n(x) = \sum_{a=-n/2}^{n/2} (x + ia)^L \frac{Q_1(x + i(n + 1)/2)Q_2(x + ia)}{Q_1(x + i(a + 1/2))Q_1(x + i(a - 1/2))}.
$$

The validity of the $\mathfrak{su}(3)$ formula has been checked numerically for a plethora of $\mathfrak{su}(3)$ states.
The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k (\{u_j\}) = \left[ Q_1 (0) Q_1 (i/2) \cdot \frac{\det G^+}{\det G^-} \right]^{1/2} \cdot \sum_{a=(1-k)/2}^{(k-1)/2} \frac{a^L Q_1 (ik/2)}{Q_1 (i(a+1/2))Q_1 (i(a-1/2))}.$$ 

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k (\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1 (i/2)}{Q_1 (0)} \frac{Q_2^2 (i/2)}{R_2 (0) R_2 (i/2)} \cdot \frac{\det G^+}{\det G^-}},$$

de Leeuw-Kristjansen-Mori, 2016

where,

$$\phi_{1,i} = -i \log \left[ \left( \frac{u_{1,i} - i/2}{u_{1,i} + i/2} \right)^L \prod_{j \neq i}^{N_1} \frac{u_{1,i} - u_{1,j} + i}{u_{1,i} - u_{1,j} - i} \prod_{k=1}^{N_2} \frac{u_{1,i} - u_{2,k} - i/2}{u_{1,i} - u_{2,k} + i/2} \prod_{l=1}^{N_3} \frac{u_{1,i} - u_{3,l} - i/2}{u_{1,i} - u_{3,l} + i/2} \right]$$

$$\phi_{2,i} = -i \log \left[ \prod_{l \neq i}^{N_2} \frac{u_{2,i} - u_{2,l} + i}{u_{2,i} - u_{2,l} - i} \prod_{k=1}^{N_1} \frac{u_{2,i} - u_{1,k} - i/2}{u_{2,i} - u_{1,k} + i/2} \right].$$
The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k (\{u_j\}) = \left[ Q_1 (0) Q_1 (i/2) \cdot \frac{\det G^+}{\det G^-} \right]^{1/2} \cdot \sum_{a=(1-k)/2}^{(k-1)/2} \frac{a^L Q_1 (ik/2)}{Q_1 (i(a+1/2)) Q_1 (i(a-1/2))},$$

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k (\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1 (i/2)}{Q_1 (0)}} \frac{Q_2 (i/2)}{R_2 (0) R_2 (i/2)} \cdot \frac{\det G^+}{\det G^-}.$$

de Leeuw-Kristjansen-Mori, 2016

For $A_\pm = A_1 \pm A_2$, $B_\pm = B_1 \pm B_2$, $C_\pm = C_1 \pm C_2$, we define:

$$G \equiv \frac{\partial \phi_I}{\partial u_j} = \begin{bmatrix}
A_1 & A_2 & B_1 & B_2 & D_1 \\
A_2 & A_1 & B_2 & B_1 & D_1 \\
B_1^t & B_2^t & C_1 & C_2 & D_2 \\
B_2^t & B_1^t & C_2 & C_1 & D_2 \\
D_1^t & D_1^t & D_2^t & D_2^t & D_3
\end{bmatrix}, \quad G^+ = \begin{pmatrix} A_+ & B_+ & D_1 \\
B_+^t & C_+ & D_2 \\
2D_1^t & 2D_2^t & D_3
\end{pmatrix}, \quad G^- = \begin{pmatrix} A_- & B_- \\
B_-^t & C_-
\end{pmatrix}.$$
The $\mathfrak{su}(3)$ determinant formula

For $N_+ = 0$ the $\mathfrak{su}(3)$ formula reduces to the $\mathfrak{su}(2)$ formula that we saw before:

$$C_k (\{u_j\}) = \left[ Q_1 (0) Q_1 (i/2) \cdot \frac{\det G^+}{\det G^-} \right]^{1/2} \cdot \left( \frac{(k-1)/2}{a=(1-k)/2} \sum \frac{a^L Q_1(ik/2)}{Q_1(i(a+1/2))Q_1(i(a-1/2))} \right).$$

For $k = 2$ it reduces to a known $\mathfrak{su}(3)$ formula:

$$C_k (\{u_j; v_j\}) = 2^{1-L} \cdot \sqrt{\frac{Q_1(i/2)}{Q_1(0)}} \frac{Q_2^2(i/2)}{R_2(0) R_2(i/2)} \cdot \frac{\det G^+}{\det G^-}. $$

de Leeuw-Kristjansen-Mori, 2016

Here are some more properties of one-point functions in $\mathfrak{su}(3)$:

- One-point functions vanish if $M$ or $L + N_+$ is odd.

- Because $Q_3 \cdot |\text{MPS}\rangle = 0$ all 1-point functions vanish unless all the Bethe roots are fully balanced:

$$\{ u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0 \}, \quad \{ v_1, \ldots, v_{N_+/2}, -v_1, \ldots, -v_{N_+/2}, 0 \}. $$
The $\mathfrak{so}(6)$ determinant formula

The one-point function in the $\mathfrak{so}(6)$ sector is given by

$$C_k (\{u_j; v_j; w_j\}) = T_{k-1}(0) \cdot \sqrt{\frac{Q_1(0) Q_1(i/2) Q_1(ik/2) Q_1(ik/2)}{R_2(0) R_2(i/2) R_3(0) R_3(i/2)}} \cdot \frac{\det G^+}{\det G^-}$$

where $u_i \equiv u_{1,i}$, $v_j \equiv u_{2,j}$, $w_k \equiv u_{3,k}$ and

$$T_n(x) = \sum_{a=-n/2}^{n/2} (x + ia)^L \frac{Q_2(x + ia) Q_3(x + ia)}{Q_1(x + i(a + 1/2)) Q_1(x + i(a - 1/2))}.$$  

More properties of one-point functions in $\mathfrak{so}(6)$:

- One-point functions vanish if $M$ or $L + N_+ + N_-$ is odd.
- Because $Q_3 \cdot |\text{MPS}\rangle = 0$, all 1-point functions vanish unless all the Bethe roots are fully balanced:

$$\{u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0\}$$

$$\{v_1, \ldots, v_{N_+/2}, -v_1, \ldots, -v_{N_+/2}, 0\}, \quad \{w_1, \ldots, w_{N_-/2}, -w_1, \ldots, -w_{N_-/2}, 0\}.$$
The $so(6)$ determinant formula

The norm matrix is defined as follows:

$$
G \equiv \partial_J \phi_I = \frac{\partial \phi_I}{\partial u_J} = 
\begin{pmatrix}
A_1 & A_2 & B_1 & B_2 & D_1 & F_1 & F_2 & H_1 \\
A_2 & A_1 & B_2 & B_1 & D_2 & F_2 & F_1 & H_1 \\
B_1 & B_2 & C_1 & C_2 & D_2 & K_1 & K_2 & H_2 \\
B_2 & B_1 & C_2 & C_1 & D_2 & K_2 & K_1 & H_2 \\
D_1^1 & D_1^2 & D_2^1 & D_2^2 & D_3 & D_3 & D_4 & D_4^1 \\
F_1^1 & F_1^2 & K_1^1 & K_1^2 & D_4 & L_1 & L_2 & H_4 \\
F_2^1 & F_2^2 & K_2^1 & K_2^2 & D_4 & L_2 & L_1 & H_4 \\
H_1 & H_2 & H_3 & H_4 & H_5 &
\end{pmatrix},
$$

where

$$
\phi_I \equiv \{\phi_{1,i}, \phi_{2,j}, \phi_{3,k}\}, \quad i = 1, \ldots, N_1, \quad j = 1, \ldots, N_2, \quad k = 1, \ldots, N_3
$$

$$
u_J \equiv \{u_{1,i}, u_{2,j}, u_{3,k}\}, \quad I, J = 1, \ldots, N_1 + N_2 + N_3,$n

and

$$
\phi_{1,i} = -i \log \left[ \left( \frac{u_{1,i} - i/2}{u_{1,i} + i/2} \right) \prod_{j \neq i}^{N_1} \frac{u_{1,i} - u_{1,j} + i}{u_{1,i} - u_{1,j} - i} \prod_{k=1}^{N_2} \frac{u_{1,i} - u_{2,k} - i/2}{u_{1,i} - u_{2,k} + i/2} \prod_{l=1}^{N_3} \frac{u_{1,i} - u_{3,l} - i/2}{u_{1,i} - u_{3,l} + i/2} \right],
$$

$$
\phi_{2,i} = -i \log \left[ \prod_{l \neq i}^{N_2} \frac{u_{2,i} - u_{2,l} + i}{u_{2,i} - u_{2,l} - i} \prod_{k=1}^{N_2} \frac{u_{2,i} - u_{1,k} - i/2}{u_{2,i} - u_{1,k} + i/2} \right], \quad \phi_{3,i} = -i \log \left[ \prod_{l \neq i}^{N_3} \frac{u_{3,i} - u_{3,l} + i}{u_{3,i} - u_{3,l} - i} \prod_{k=1}^{N_3} \frac{u_{3,i} - u_{1,k} - i/2}{u_{3,i} - u_{1,k} + i/2} \right].
$$
The so(6) determinant formula

It can be shown that the determinant of the norm matrix factorizes:

\[ \det G = \det G_+ \cdot \det G_- , \]

with \( A_\pm \equiv A_1 \pm A_2 \) (and so on for \( B_\pm, C_\pm, F_\pm, K_\pm, L_\pm \)), while

\[
G_+ = \begin{pmatrix}
A_+ & B_+ & D_1 & F_+ & H_1 \\
B^\top_+ & C_+ & D_2 & K_+ & H_2 \\
2D^\top_1 & 2D^\top_2 & D_3 & 2D^\top_4 & H_3 \\
F^\top_+ & K^\top_+ & D_4 & L_+ & H_4 \\
2H^\top_1 & 2H^\top_2 & 2H^\top_3 & 2H^\top_4 & H_5
\end{pmatrix}
\]

& \quad G_- = \begin{pmatrix}
A_- & B_- & F_- \\
B^\top_- & C_- & K_- \\
F^\top_- & K^\top_- & L_- \\
\end{pmatrix} .

An unproven claim (Escobedo, 2012) is that the norm of any so(6) Bethe eigenstate is given by the determinant of its norm matrix:

\[ n(L, N_1, N_2, N_3) = \det G . \]
Section 2

One-point Functions in the D3-D7 System

The \( \text{SO}(5) \) symmetric D3-D7 system: description

- In the bulk, the D3-D7 system describes IIB superstring theory on \( \text{AdS}_5 \times \text{S}^5 \) bisected by a D7-brane with worldvolume geometry \( \text{AdS}_4 \times \text{S}^4 \).

- The dual field theory is still \( SU(N), \mathcal{N} = 4 \) SYM in 3 + 1 dimensions, that interacts with a CFT living on the 2 + 1 dimensional defect:
  \[
  S = S_{\mathcal{N}=4} + S_{2+1}.
  \]

- Due to the presence of the defect, the total bosonic symmetry of the system is reduced from \( \text{SO}(4, 2) \times \text{SO}(6) \) to \( \text{SO}(3, 2) \times \text{SO}(5) \).

- The relative co-dimension of the branes is \#ND = 6 \( \rightarrow \) no unbroken supersymmetry.

- Tachyonic instability...
The $(D3-D7)_{d_G}$ system

- To stabilize the system, add an instanton bundle on the $S^4$ component of the $AdS_4 \times S^4$ D7-brane, with instanton number $d_G = (n+1)(n+2)(n+3)/6$.
  (Myers-Wapler, 2008)

- Then exactly $d_G$ of the $N$ D3-branes ($N \gg d_G$) will end on the D7-brane.

- On the dual gauge theory side, the gauge group $SU(N) \times SU(N)$ breaks to $SU(N) \times SU(N - d_G)$.

- Equivalently, the fields of $\mathcal{N} = 4$ SYM develop nonzero vevs...
  (Karch-Randall, 2001b)
Subsection 2

Nested one-point functions at tree-level
The dCFT interface of D3-D7

- As before, we need an interface to separate the $SU(N)$ and $SU(N - d_G)$ regions of the (D3-D7)$_{d_G}$ dCFT...

- For no vectors/fermions, we want to solve the equations of motion for the scalar fields of $\mathcal{N} = 4$ SYM:
  \[
  A_\mu = \psi_a = 0, \quad \frac{d^2 \Phi_i}{dz^2} = [\Phi_j, [\Phi_j, \Phi_i]], \quad i,j = 1, \ldots, 6.
  \]

- A manifestly $SO(5) \subset SO(3,2) \times SO(5)$ symmetric solution is given by $(z > 0)$:
  \[
  \Phi_i(z) = \frac{G_i \oplus 0_{(N-d_G) \times (N-d_G)}}{\sqrt{8z}}, \quad i = 1, \ldots, 5, \quad \Phi_6 = 0.
  \]

Kristjansen-Semenoff-Young, 2012

The matrices $G_i$ are known as "fuzzy" $S^4$ matrices or "G-matrices".
The "fuzzy" $S^4$ $G$-matrices

Here's the definition of the five $d_G \times d_G$ "fuzzy" $S^4$ matrices ($G$-matrices) $G_i$:

$$G_i \equiv \left[ \begin{array}{c}
\gamma_i \otimes 1_{4 \otimes \ldots \otimes 1_{4}} + 1_{4} \otimes \gamma_i \otimes \ldots \otimes 1_{4} + \ldots + 1_{4} \otimes \ldots \otimes 1_{4} \otimes \gamma_i \\
\end{array} \right]_{\text{sym}}$$

(i = 1, \ldots, 5),

Castelino-Lee-Taylor, 1997

where $\gamma_i$ are the five $4 \times 4$ Euclidean Dirac matrices:

$$\gamma_i = \left( \begin{array}{cc}
0 & -i \sigma_i \\
n_i \sigma_i & 0
\end{array} \right), \quad i = 1, 2, 3, \quad \gamma_4 = \left( \begin{array}{cc}
0 & 1_2 \\
1_2 & 0
\end{array} \right), \quad \gamma_5 = \left( \begin{array}{cc}
1_2 & 0 \\
0 & -1_2
\end{array} \right)$$

and $\sigma_i$ are the three $2 \times 2$ Pauli matrices. The ten commutators of the five $G$-matrices,

$$G_{ij} \equiv \frac{1}{2} [G_i, G_j]$$

furnish a $d_G$-dimensional (anti-hermitian) irreducible representation of $so(5) \simeq sp(4)$:

$$[G_{ij}, G_{kl}] = 2 (\delta_{jk} G_{il} + \delta_{il} G_{jk} - \delta_{ik} G_{jl} - \delta_{jl} G_{ik}).$$
The "fuzzy" $S^4$ G-matrices

The dimension of the G-matrices is equal to the instanton number $d_G = (n + 1)(n + 2)(n + 3)/6$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_G$</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>56</td>
<td>84</td>
<td>120</td>
<td>165</td>
<td>220</td>
<td>286</td>
<td>...</td>
</tr>
</tbody>
</table>

E.g., for $n = 2$, here are the $10 \times 10$ G-matrices:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$G_5 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
1-point functions

The 1-point functions of local gauge-invariant scalar operators

\[ \langle O(z, x) \rangle = \frac{C}{z^\Delta}, \quad z > 0, \]

can again be calculated within the D3-D7 dCFT from the corresponding fuzzy-funnel solution, e.g.

\[ O(z, x) = \Psi_{i_1 \ldots i_L} \text{Tr} [\Phi_{i_1} \ldots \Phi_{i_L}] \xrightarrow{\text{SO}(5)\text{ interface}} \frac{1}{8^{L/2} z^L} \cdot \Psi_{i_1 \ldots i_L} \text{Tr} [G_{i_1} \ldots G_{i_L}] \]

where \( \Psi_{i_1 \ldots i_L} \) is an \( \mathfrak{so}(6) \)-symmetric tensor and the constant \( C \) is given by (MPS=matrix product state)

\[ C = \frac{1}{\sqrt{L}} \left( \frac{\pi^2}{\lambda} \right)^{L/2} \frac{\langle \text{MPS} | \Psi \rangle}{\langle \Psi | \Psi \rangle^{1/2}}, \quad \begin{cases} \langle \text{MPS} | \Psi \rangle \equiv \Psi_{i_1 \ldots i_L} \text{Tr} [G_{i_1} \ldots G_{i_L}] \quad ("overlap") \\ \langle \Psi | \Psi \rangle \equiv \Psi_{i_1 \ldots i_L} \Psi_{i_1 \ldots i_L} \end{cases} \]

- The mixing of single-trace operators up to one-loop in \( \mathcal{N} = 4 \) SYM is described by the integrable \( \mathfrak{so}(6) \) spin chain of Minahan-Zarembo.
- We will assume that the above result is unaffected in the dCFT that is dual to the D3-D7 system.
Example: chiral primary operators

The one-point function of the chiral primary operators

$$\mathcal{O}_{\text{CPO}}(x) = \frac{1}{\sqrt{L}} \left( \frac{8\pi^2}{\lambda} \right)^{L/2} \cdot C^{i_1 \ldots i_L} \text{Tr} [\Phi_{i_1} \ldots \Phi_{i_L}],$$

where $C^{i_1 \ldots i_L}$ are symmetric & traceless tensors satisfying

$$C^{i_1 \ldots i_L} C^{j_1 \ldots j_L} = 1 \quad \& \quad Y_L = C^{i_1 \ldots i_L} \hat{x}_{i_1} \ldots \hat{x}_{i_L}, \quad \sum_{i=4}^{9} \hat{x}_i^2 = 1,$$

and $Y_L(\theta)$ is the $SO(5)$ spherical harmonic ($Y_{\text{odd}}(0) = 0$), have been calculated at weak coupling:

$$\langle \mathcal{O}_{\text{CPO}}(x) \rangle = \frac{d_G}{\sqrt{L}} \left( \frac{\pi^2 c_G}{\lambda} \right)^{L/2} \frac{Y_L(0)}{z^L}, \quad c_G \equiv n(n + 4), \quad d_G \ll N \to \infty.$$

Kristjansen-Semenoff-Young, 2012

The large-$n$ limit reproduces the supergravity calculation:

$$\langle \mathcal{O}_{\text{CPO}}(x) \rangle \xrightarrow{n \to \infty} \frac{Y_L(0)}{\sqrt{L}} \left( \frac{\pi^2}{\lambda} \right)^{L/2} \frac{n^3}{z^L}.$$
Bethe state overlaps

- The matrix product state projects the 3 complex scalars on the \( SO(5) \) fuzzy funnel solution:

\[
\langle \text{MPS} | \Psi \rangle = z^L \cdot \sum_{1 \leq x_k \leq L} \psi(x_k) \cdot \text{Tr} \left[ Z^{x_1-1} W Z^{x_2-x_1-1} Y Z^{x_3-x_2-1} W Z^{x_4-x_3-1} Y \ldots \right]
\]

where the complex scalar fields \( Z, W, Y \) are expressed in terms of the \( G \)-matrices as follows:

- \( W \sim G_1 + iG_2 \)
- \( Y \sim G_3 + iG_4 \)
- \( Z \sim G_5 \)

- \( \bar{W} \sim G_1 - iG_2 \)
- \( \bar{Y} \sim G_3 - iG_4 \)
- \( \bar{Z} \sim G_5 \)

- The corresponding matrix product state (MPS) is given by:

\[
|\text{MPS}\rangle = \text{Tr}_a \left[ \prod_{l=1}^{L} \left( |Z\rangle_l \otimes G_5 \right) + |W\rangle_1 \otimes (G_1 + iG_2) + |Y\rangle_1 \otimes (G_3 + iG_4) + \text{c.c.} \right].
\]

It can be proven that all possible assignments for the fields \( Z, W, Y \) are equivalent.
Subsection 3

Determinant formulas
SO(5) vacuum overlap

For the vacuum overlap we have found:

\[ \langle \text{MPS}|0 \rangle = \text{Tr} \left[ G_5^L \right] = \sum_{j=1}^{n+1} j (n - j + 2) (n - 2j + 2)^L. \]

Changing variables \( j \leftrightarrow (n + 2 - j) \), an overall factor \((-1)^L\) comes out, leading the vacuum overlap to zero for \( L \) odd. Equivalently, we may write

\[ \langle \text{MPS}|0 \rangle = 2^L \left[ \frac{(n + 2)^2}{4} \left( \zeta \left( -L, -\frac{n}{2} \right) - \zeta \left( -L, \frac{n}{2} + 1 \right) \right) - \left( \zeta \left( -L - 2, -\frac{n}{2} \right) - \zeta \left( -L - 2, \frac{n}{2} + 1 \right) \right) \right], \]

where the Hurwitz zeta function is defined as:

\[ \zeta (s, a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}. \]
SO(5) vacuum overlap

For the vacuum overlap we have found:

\[ \langle \text{MPS}|0 \rangle = \text{Tr} \left[ G_5^L \right] = \sum_{j=1}^{n+1} j (n - j + 2) (n - 2j + 2)^L. \]

Changing variables \( j \leftrightarrow (n + 2 - j) \), an overall factor \((-1)^L\) comes out, leading the vacuum overlap to zero for \( L \) odd. Equivalently, we may write

\[ \langle \text{MPS}|0 \rangle = \begin{cases} 
0, & L \text{ odd} \\
2^L \cdot \left[ \frac{2}{L+3} B_{L+3} \left(-\frac{n}{2}\right) - \frac{(n+2)^2}{2(L+1)} B_{L+1} \left(-\frac{n}{2}\right) \right], & L \text{ even,} 
\end{cases} \]

by using the relationship between the Hurwitz zeta function and the Bernoulli polynomials \( B_m(x) \).
For the vacuum overlap we have found:

$$\langle \text{MPS}|0 \rangle = \text{Tr} \left[ G^L_5 \right] = \sum_{j=1}^{n+1} \left[ j \left( n - j + 2 \right) \left( n - 2j + 2 \right)^L \right].$$

Changing variables $j \leftrightarrow (n + 2 - j)$, an overall factor $(-1)^L$ comes out, leading the vacuum overlap to zero for $L$ odd. Equivalently, we may write

$$\langle \text{MPS}|0 \rangle = \begin{cases} 0, & L \text{ odd} \\ 2^L \cdot \left[ \frac{2}{L+3} B_{L+3} \left( -\frac{n}{2} \right) - \frac{(n+2)^2}{2(L+1)} B_{L+1} \left( -\frac{n}{2} \right) \right], & L \text{ even,} \end{cases}$$

by using the relationship between the Hurwitz zeta function and the Bernoulli polynomials $B_m(x)$. In the large-$n$ limit we find:

$$\langle \text{MPS}|0 \rangle \sim \frac{n^{L+3}}{2(L+1)(L+3)} + O \left( n^{L+2} \right), \quad n \to \infty.$$
Overlap properties

- The overlaps $\langle \text{MPS}|\Psi \rangle$ of all the highest-weight eigenstates vanish unless:

$$\# \mathcal{W} = \# \overline{\mathcal{W}}, \quad \# \mathcal{Y} = \# \overline{\mathcal{Y}}.$$ 

Therefore the only $\mathfrak{so}(6)$ eigenstates that have nonzero one-point functions are those with:

$$N_1 = 2N_2 = 2N_3 \equiv M \ (\text{even}).$$

Evidently, all one-point functions vanish in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors.
**Overlap properties**

- The overlaps $\langle \text{MPS} | \Psi \rangle$ of all the highest-weight eigenstates vanish unless:
  
  $$\# \mathcal{W} = \# \overline{\mathcal{W}}, \quad \# \mathcal{V} = \# \overline{\mathcal{V}}.$$ 

  Therefore the only $\mathfrak{so}(6)$ eigenstates that have nonzero one-point functions are those with:
  
  $$N_1 = 2N_2 = 2N_3 \equiv M \ (\text{even}).$$

  Evidently, all one-point functions vanish in the $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ subsectors.

- Because the third conserved charge $Q_3$ annihilates the matrix product state:
  
  $$Q_3 \cdot |\text{MPS}\rangle = 0,$$

  all the one-point functions will vanish, unless all the Bethe roots are fully balanced:
  
  $$\{u_1, \ldots, u_{M/2}, -u_1, \ldots, -u_{M/2}, 0\},$$
  
  $$\{v_1, \ldots, v_{N+/2}, -v_1, \ldots, -v_{N+/2}, 0\},$$
  
  $$\{w_1, \ldots, w_{N-/2}, -w_1, \ldots, -w_{N-/2}, 0\}.$$
Example: the Konishi operator

- A prime example of a non-protected operator is the Konishi operator:

\[ \mathcal{K} = \text{Tr} [\Phi_i \Phi_i] = \text{Tr} [Z \bar{Z}] + \text{Tr} [W \bar{W}] + \text{Tr} [\mathcal{Y} \bar{\mathcal{Y}}] \]

which is an eigenstate of the \( so(6) \) Hamiltonian with \( L = N_1 = 2, N_2 = N_3 = 1 \) and eigenvalue:

\[ E = 2 + \frac{3\lambda}{4\pi^2} + \ldots \]

- Using the Casimir relation:

\[ \text{Tr} [G_i G_i] = \frac{1}{6} n (n + 1) (n + 2) (n + 3) (n + 4) \]

we can compute the one-point function of the Konishi operator:

\[ \langle \mathcal{K} \rangle = \frac{1}{6\sqrt{3}} \frac{\pi^2}{\lambda} n (n + 1) (n + 2) (n + 3) (n + 4). \]
The $L211$ states

- More generally, we can consider eigenstates with $N_1 = 2$, $N_2 = N_3 = 1$ and arbitrary $L$:

$$|p⟩ = \sum_{x_1 < x_2} \left( e^{ip(x_1 - x_2)} + e^{ip(x_2 - x_1 + 1)} \right) · |...\chi_{x_1}...\chi_{x_2}...⟩ - 2 \sum_{x_3} (1 + e^{ip}) · |...\bar{Z}_{x_3}...⟩,$$

where the dots stand for $\bar{Z}$, and $\chi$ is any of the complex scalars $\mathcal{W}$, $\bar{\mathcal{W}}$, $\mathcal{Y}$, $\bar{\mathcal{Y}}$.

- The momentum $p$ is found by solving the corresponding Bethe equations:

$$e^{ip(L+1)} = 1 \Rightarrow p = \frac{4m\pi}{L+1}, \quad m = 1, \ldots, L + 1$$

- Here’s the one-loop energy of the $L211$ eigenstates:

$$E = L + \frac{\lambda}{\pi^2} \sin^2 \left[ \frac{2m\pi}{L+1} \right] + \ldots, \quad m = 1, \ldots, L + 1$$
The $L211$ determinant formula

- The corresponding one-point function for all $n$ is given in terms of the $n = 1$ one:

$$
\langle O_{L211} \rangle = \left[ \frac{u^2}{u^2 - 1/2} \sum_{n \mod 2}^n j^L \cdot \frac{(n + 2)^2 - j^2}{8} \cdot \left[ u^2 + \frac{(n+2)j+1}{4}u^2 - \frac{(n+2)j-1}{4} \right] \right] \cdot \langle O_{L211}^{n=1} \rangle
$$

where

$$
\langle O_{L211}^{n=1} \rangle = 8 \sqrt{\frac{L}{L+1}} \frac{u^2 - \frac{1}{2}}{u^2 + \frac{1}{4}} \sqrt{\frac{u^2 + \frac{1}{4}}{u^2}}, \quad u \equiv \frac{1}{2} \cot \frac{p}{2}.
$$

- The results fully reproduce the numerical values (given in units of $(\pi^2/\lambda)^{L/2}/\sqrt{L}$):

<table>
<thead>
<tr>
<th>$L$</th>
<th>$N_{1/2/3}$</th>
<th>eigenvalue $\gamma$</th>
<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
<th>$n=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 1 1</td>
<td>6</td>
<td>$20\sqrt{\frac{2}{3}}$</td>
<td>$40\sqrt{6}$</td>
<td>$140\sqrt{6}$</td>
<td>$1120\sqrt{2}$</td>
</tr>
<tr>
<td>4</td>
<td>2 1 1</td>
<td>$5 + \sqrt{5}$</td>
<td>$20 + \frac{44}{\sqrt{5}}$</td>
<td>$96\left(15 + \sqrt{5}\right)$</td>
<td>$84\left(21 - \sqrt{5}\right)$</td>
<td>$\frac{3584}{5}\left(10 - \sqrt{5}\right)$</td>
</tr>
<tr>
<td>4</td>
<td>2 1 1</td>
<td>$5 - \sqrt{5}$</td>
<td>$20 - \frac{44}{\sqrt{5}}$</td>
<td>$288 - \frac{96}{\sqrt{5}}$</td>
<td>$84\left(21 + \sqrt{5}\right)$</td>
<td>$\frac{3584}{5}\left(10 + \sqrt{5}\right)$</td>
</tr>
<tr>
<td>6</td>
<td>2 1 1</td>
<td>1.50604</td>
<td>3.57792</td>
<td>324.178</td>
<td>11338.3</td>
<td>98726</td>
</tr>
<tr>
<td>6</td>
<td>2 1 1</td>
<td>4.89008</td>
<td>9.90466</td>
<td>1724.55</td>
<td>19513.8</td>
<td>120347</td>
</tr>
<tr>
<td>6</td>
<td>2 1 1</td>
<td>7.60388</td>
<td>61.6252</td>
<td>1044.86</td>
<td>8830.95</td>
<td>49114.4</td>
</tr>
</tbody>
</table>
The $L422$ determinant formula

For the eigenstates with $N_1 = 4, N_2 = N_3 = 2$ and $L = \text{even}$, we find (work in progress):

$$\langle O_{L422} \rangle = \sum_{n \mod 2} \frac{n^L}{4} \cdot \frac{(n+2)^2 - j^2}{4} \cdot \frac{Q_1 \left[ i \sqrt{(n+2)j+1} \right]}{Q_1 \left[ i (j-\frac{1}{2}) \right]} \cdot \left[ (1 + (-1)^L) \cdot Q_1 \left[ \frac{\sqrt{(n+2)j-1}}{2} \right] - (5n-2) \left( Q_2[0] + (-1)^L Q_3[0] \right) \right] \sqrt{G^+ G^-}$$

The results fully reproduce the corresponding numerical values for $n = 1$ and $n = 2$:

<table>
<thead>
<tr>
<th>$L$</th>
<th>$N_{1/2/3}$</th>
<th>eigenvalue $\gamma$</th>
<th>$n=1$</th>
<th>$n=2$</th>
<th>$n=3$</th>
<th>$n=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4 2 2</td>
<td>$\frac{1}{2} \left( 13 + \sqrt{41} \right)$</td>
<td>2$\sqrt{1410 + \frac{25970}{3\sqrt{41}}} \cdot 16 \sqrt{3090 + \frac{10710}{\sqrt{41}}} \cdot 14 \sqrt{161490 + \frac{140310}{\sqrt{41}}} \cdot 896 \sqrt{690 - \frac{670}{3\sqrt{41}}}$</td>
<td>2$\sqrt{1410 - \frac{25970}{3\sqrt{41}}} \cdot 16 \sqrt{3090 - \frac{10710}{\sqrt{41}}} \cdot 14 \sqrt{161490 - \frac{140310}{\sqrt{41}}} \cdot 896 \sqrt{690 + \frac{670}{3\sqrt{41}}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4 2 2</td>
<td>$\frac{1}{2} \left( 13 - \sqrt{41} \right)$</td>
<td>2$\sqrt{1410 + \frac{25970}{3\sqrt{41}}} \cdot 16 \sqrt{3090 - \frac{10710}{\sqrt{41}}} \cdot 14 \sqrt{161490 + \frac{140310}{\sqrt{41}}} \cdot 896 \sqrt{690 - \frac{670}{3\sqrt{41}}}$</td>
<td>2$\sqrt{1410 - \frac{25970}{3\sqrt{41}}} \cdot 16 \sqrt{3090 + \frac{10710}{\sqrt{41}}} \cdot 14 \sqrt{161490 - \frac{140310}{\sqrt{41}}} \cdot 896 \sqrt{690 + \frac{670}{3\sqrt{41}}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>8</td>
<td>4.76832</td>
<td>2899.14</td>
<td>37483.7</td>
<td>247800</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>2.26228</td>
<td>8.68876</td>
<td>1090.46</td>
<td>11963</td>
<td>166654</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>3.81374</td>
<td>13.8862</td>
<td>4479.21</td>
<td>43679.9</td>
<td>238186</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>5.33676</td>
<td>22.5105</td>
<td>2995.7</td>
<td>34577.8</td>
<td>216443</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>8.94875</td>
<td>78.0614</td>
<td>1813.66</td>
<td>16647.9</td>
<td>95264.6</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>10.1954</td>
<td>138.297</td>
<td>151.877</td>
<td>10250</td>
<td>80604.6</td>
</tr>
<tr>
<td>6</td>
<td>4 2 2</td>
<td>12.4431</td>
<td>369.992</td>
<td>4881.61</td>
<td>33331.2</td>
<td>159221</td>
</tr>
</tbody>
</table>

The $L422$ formula reduces to the previous one for $N_1 = 2, N_2 = N_3 = 1$. 
Section 3

Summary
Summary

We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric $(D3-D5)_k$ dCFT and the $SO(5)$ symmetric $(D3-D7)_{d_G}$ dCFT...
Summary

We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric (D3-D5)$_k$ dCFT and the $SO(5)$ symmetric (D3-D7)$_{d_G}$ dCFT...

D3-D5 dCFT

- Because $Q_3 \cdot |\text{MPS}\rangle = 0$, all 1-point functions vanish unless the Bethe roots are fully balanced:
  \[
  \{u_{1,i}\} = \{-u_{1,i}\}, \quad \{u_{2,i}\} = \{-u_{2,i}\}, \quad \{u_{3,i}\} = \{-u_{3,i}\}.
  \]

- In $su(2)$, all 1-point functions (vacuum included) vanish if $M$ or $L$ is odd.
- In $su(3)$, all 1-point functions vanish if (1) $M$ is odd or (2) $L + N_+$ is odd.
- In $so(6)$, all 1-point functions vanish if (1) $M$ is odd or (2) $L + N_+ + N_-$ is odd.
- We have found a determinant formula for the eigenstates, valid for all values of the flux $k$:
  \[
  C_k \left( \{u_j; v_j; w_j\} \right) = \mathbb{T}_{k-1}(0) \cdot \sqrt{\frac{Q_1(0) Q_1(i/2) Q_1(ik/2) Q_1(ik/2)}{R_2(0) R_2(i/2) R_3(0) R_3(i/2)}} \cdot \frac{\det G^+}{\det G^-}
  \]
Summary

We have studied the tree-level 1-point functions of Bethe eigenstates in the $SU(2)$ symmetric $(D3-D5)_k$ dCFT and the $SO(5)$ symmetric $(D3-D7)_G$ dCFT...

**D3-D7 dCFT**

- Because $Q_3 \cdot |MPS\rangle = 0$, all 1-point functions vanish unless the Bethe roots are fully balanced:
  \[
  \{u_{1,i}\} = \{-u_{1,i}\}, \quad \{u_{2,i}\} = \{-u_{2,i}\}, \quad \{u_{3,i}\} = \{-u_{3,i}\}.
  \]

- Besides the vacuum, all 1-pt functions vanish in the $su(2)$ and $su(3)$ subsectors.

- In $so(6)$ all 1-point functions vanish unless $N_1 = 2N_2 = 2N_3 \equiv M$ (even).

- The vacuum also vanishes when $L = odd$.

- We have found a determinant formula for $L211$ eigenstates, valid for all values of the instanton number $n$:
  \[
  \langle O_{L211} \rangle = \left[ \frac{u^2}{u^2 - 1/2} \sum_{n \mod 2}^n j^L \cdot \frac{(n + 2)^2 - j^2}{8} \cdot \frac{[u^2 + \frac{(n+2)j+1}{4}][u^2 - \frac{(n+2)j-1}{4}]}{[u^2 + \frac{(j+1)^2}{2}][u^2 + \frac{(j-1)^2}{2}]} \right] \cdot \langle O_{L211}^{n=1} \rangle.
  \]
Thank you!