CORRELATORS IN N=4 SYM VIA QSC

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based on 1802.04237
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QUANTUM SPECTRAL CURVE (QSC)

See 1708.03648 for the actual introduction

XXX SPIN CHAINS

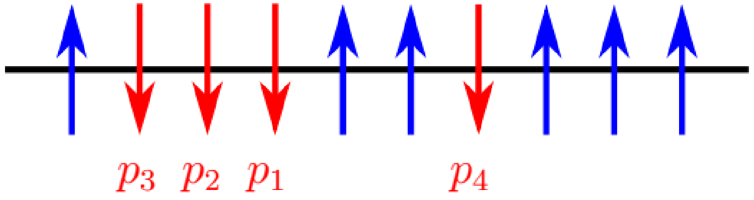
Spin chain quantization:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

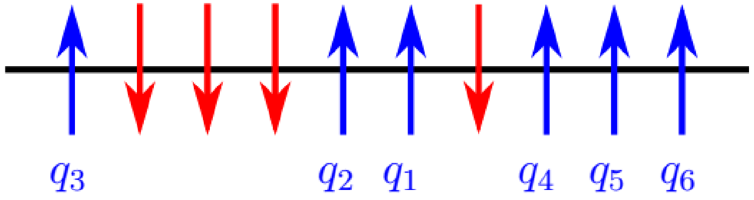
twisted polynomial

$$Q_1 \sim \lambda^{iu} u^N$$

$$Q_2 \sim \lambda^{-iu} u^{L-N}$$



L-spins, N-spins up, L-N down



Easy to generalize SU(3):

$$\begin{vmatrix} Q_1(u+i) & Q_2(u+i) & Q_3(u+i) \\ Q_1(u) & Q_2(u) & Q_3(u) \\ Q_1(u-i) & Q_2(u-i) & Q_3(u-i) \end{vmatrix} = u^L$$

GENERALIZATION TO N=4 SYM

Two main ingredients:

- QQ-relations

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

$$sl(2) \to psu(2,2|4)$$

$$(Q_1,\ Q_2) \longrightarrow (\underbrace{\mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_4}_{S^5}|\underbrace{\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3,\ \mathbf{Q}_4}_{AdS_5})$$

- Analyticity

Q_1 - Twisted polynomial

Q_2 - Twisted polynomial

In N=4 polynomials are replaced with analytic functions with cuts + monodromy condition

QQ-relations+monodromy = Quantum Spectral Curve (QSC)

Gromov,Kazakov,Leurent,Volin 2013

BAXTER EQUATION?

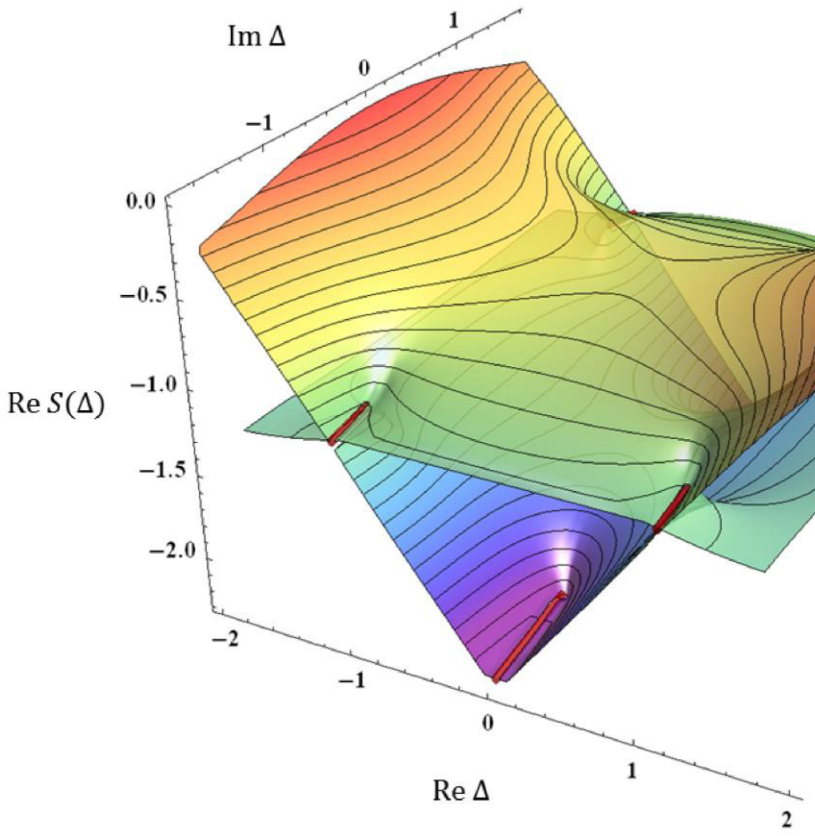
$$\mathbf{Q}_i^{[+4]} D_0 - \mathbf{Q}_i^{[+2]} \left[D_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} D_0 \right] + \mathbf{Q}_i \left[D_2 - \mathbf{P}_a \mathbf{P}^{a[+2]} D_1 + \mathbf{P}_a \mathbf{P}^{a[+4]} D_0 \right]$$
$$- \mathbf{Q}_i^{[-2]} \left[\bar{D}_1 + \mathbf{P}_a^{[-2]} \mathbf{P}^{a[-4]} \bar{D}_0 \right] + \mathbf{Q}_i^{[-4]} \bar{D}_0 = 0$$

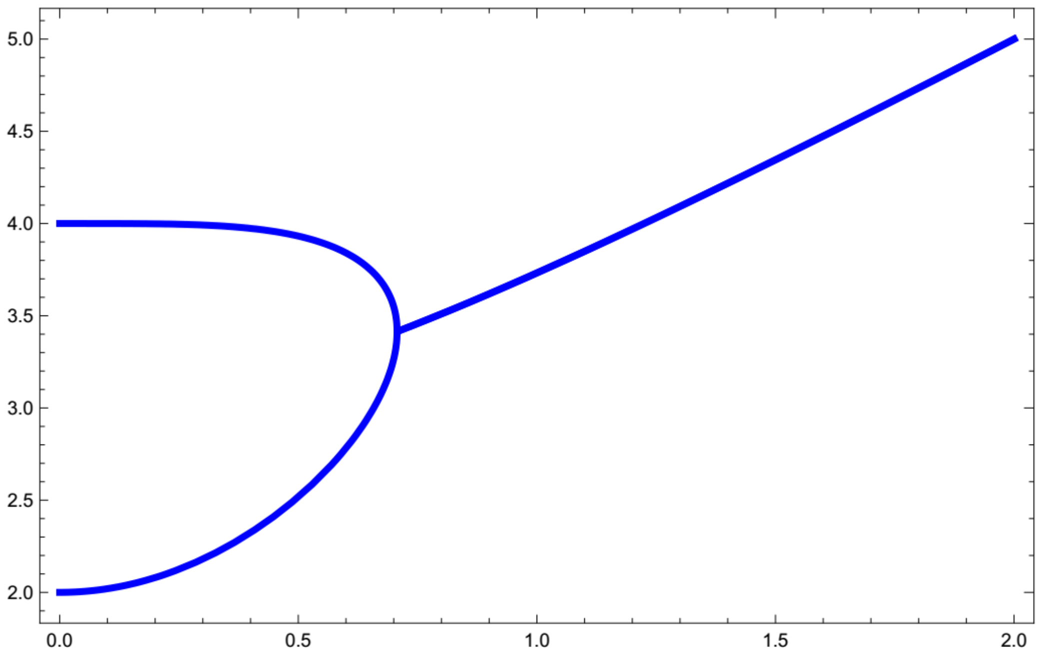
$$D_0 = \det \begin{pmatrix} \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix} \qquad D_1 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}$$

$$Q_1 \sim \lambda^{iu} u^N$$

Solve this equations => spectrum of anomalous dimensions of all local (and not only) operators

FINITE COUPLING SPECTRUM ANALYTIC STRUCTURE

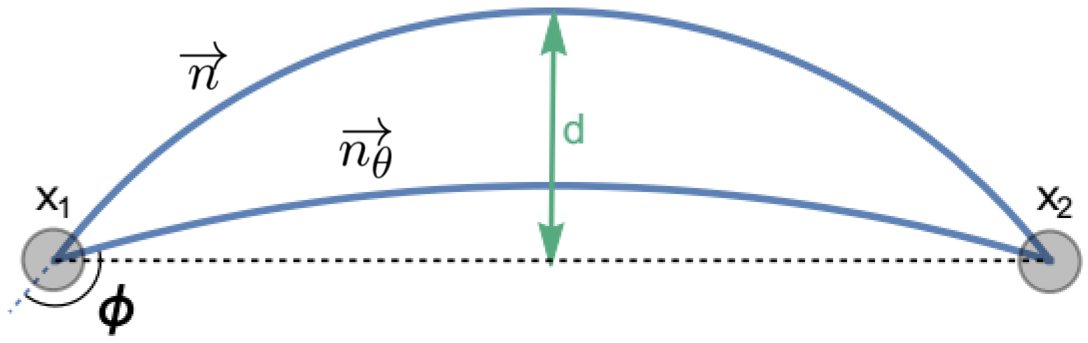




STRATEGY (SOLVING PLANAR N=4 SYM)

- We know the spectrum very well (QSC) [Gromov, Kazakov, Leurent, Volin `13]
- The result is QQ-system with PSU(2,2|4) symmetry
- The energy spectrum is an analytic multi-valued function of parameters – how can we inject these analytical properties into the structure constants?
- Inspiration comes from Spin chains – where one can explicitly build wave functions and find a basis where it is given in terms of Q-functions
- Luckily we know the Q-functions, they are part of QSC! We just need to know how to combine them into a correlation function
- Important lesson from spin chains – symmetry should be broken by twists – quasiperiodic boundary conditions. This ensures spectrum non-degenerate, Q-function is in one to one correspondence with states
- PSU(2,2|4) have to be broken, but that includes Lorentz symmetry!? We need something like Omega-background? Non-commutative theory?

CUSP IN N=4 SYM



$$W = \mathrm{Tr}\, \mathcal{P} \exp \int dt \left[iA \cdot \dot{x} + \vec{\Phi} \cdot \vec{n}\, |\dot{x}| \right]$$

Parameters:

- Cusp angle ϕ
- Angle θ between the couplings to scalars on two rays
- ‘t Hooft coupling λ

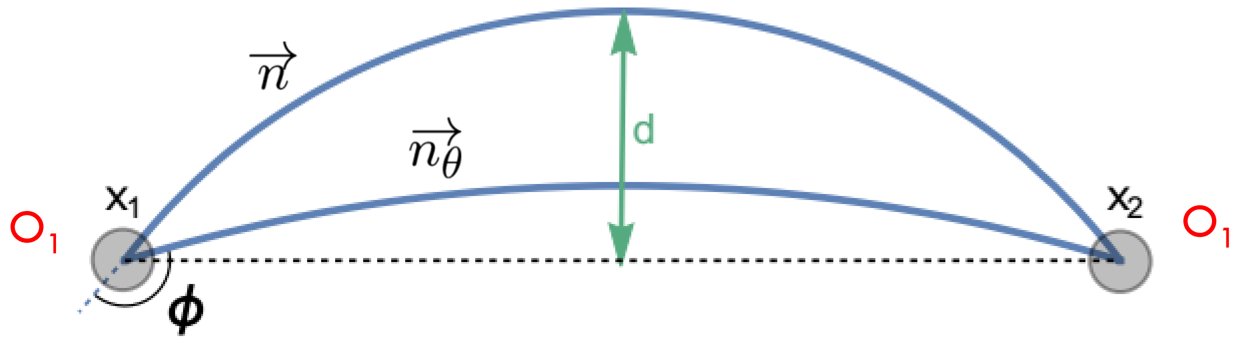
CUSP IN N=4 SYM

Drukker, Fiorini

Drukker

Maldacena, Sever, Corea

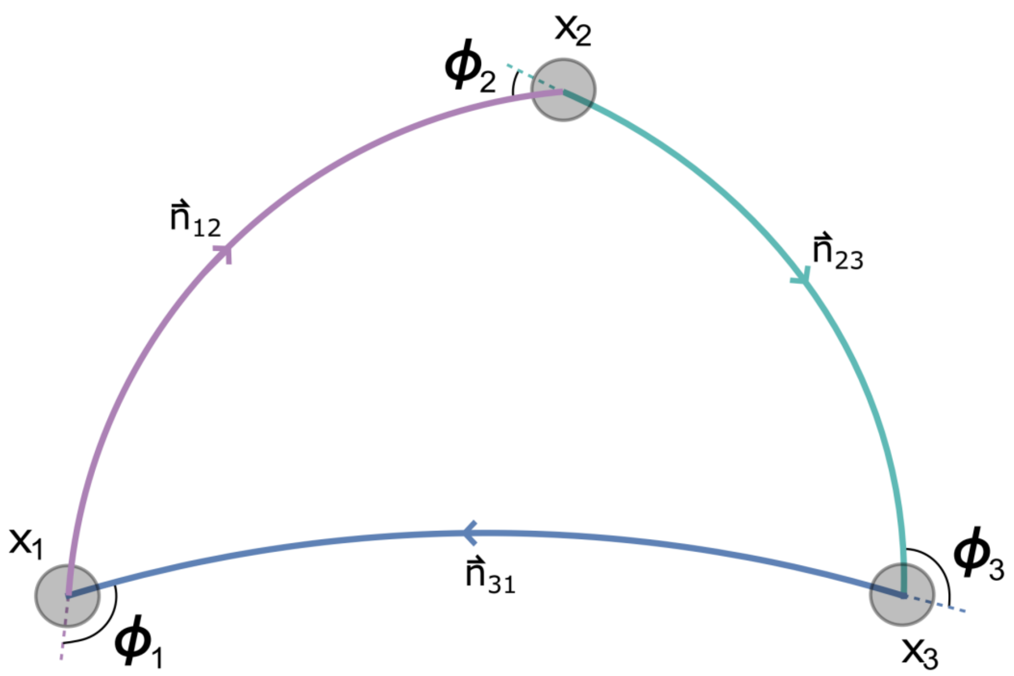
Gromov,FLM 2016



Features:

- Symmetries are twisted by WL, without spoiling the theory
- Dilatation is still here!
- Can insert any local operator at the cusp
- The dimension is governed by integrability. Cusp QSC – is very similar to that of N=4

- We got our ideal playground – explicit implementation of twists in both S5 and AdS5
- We can start deducing the structure of 3-point function in terms of Qs



- Perturbation theory?
- We follow the top fashion of the season - Fishnets

Selects only ladder diagrams, contributing to the spectrum (for some fixed phi):

$$
\begin{aligned}
\frac{\Omega}{4\pi} = \hat{g}^2 &+ \\
&\hat{g}^4 \left[16L - 8\right] + \\
&\hat{g}^6 \left[128L^2 + L\left(64 + \frac{64\pi^2 T}{3}\right) - 112 - \frac{8\pi^2}{3} + 72T\zeta_3\right] + \\
&\hat{g}^8 \left[\frac{2048L^3}{3} + \frac{1024}{3}\pi^2 L^2 T + 2048L^2 + LT\left(768\zeta_3 + \frac{2176\pi^2}{3}\right) + \left(-768 - \frac{640\pi^2}{3}\right)L \right. \\
&\left. + T^2\left(128\pi^2\zeta_3 - 760\zeta_5\right) + T\left(384\zeta_3 - 640\pi^2 + \frac{32\pi^4}{9}\right) + \frac{1664\zeta_3}{3} + \frac{1216\pi^2}{9} - 1280\right] + \ldots
\end{aligned}
$$

$$L \equiv \log\sqrt{8e^{\gamma}\pi\hat{g}^2}$$

Double scaling limit

$$T \equiv \frac{1}{\cos^2\frac{\theta}{2}}$$

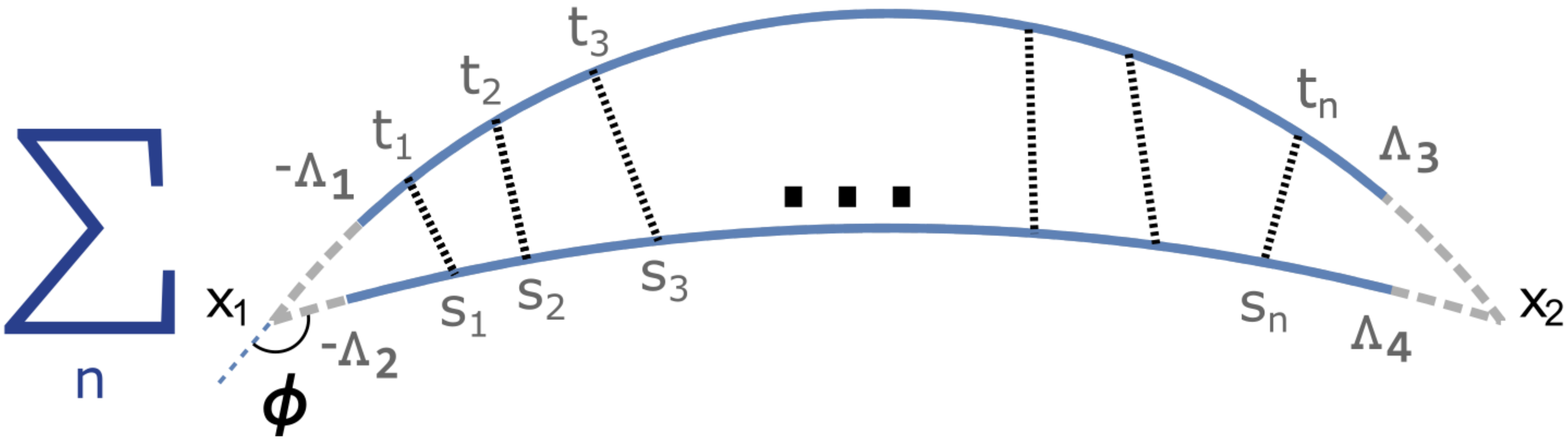
$$\hat{g} \equiv g\cos\frac{\theta}{2}$$

NG, Fedor L-M 2016

7-loop result. The term of order $\hat{g}^{14}$ in $\frac{\Omega}{4\pi}$ is given by

$$
\begin{aligned}
&\frac{1048576L^6}{45} + \frac{524288}{9}L^5\pi^2T + \frac{6815744L^5}{15} + \frac{262144}{9}L^4\pi^4T^2 - 65536L^4T\zeta_3 + \frac{40632320}{9}L^4\pi^2T \\
&-\frac{15007744}{9}L^4\pi^2 + 2752512L^4 + \frac{131072}{81}L^3\pi^6T^3 + 65536L^3\pi^2T^2\zeta_3 + \frac{655360}{3}L^3T^2\zeta_5 \\
&+\frac{12255232}{9}L^3\pi^4T^2 - \frac{64159744}{135}L^3\pi^4T - 65536L^3T\zeta_3 + \frac{13303808}{3}L^3\pi^2T + \frac{3407872L^3\zeta_3}{9} \\
&-\frac{11141120}{9}L^3\pi^2 + \frac{15073280L^3}{3} + \frac{2080768}{45}L^2\pi^4T^3\zeta_3 - \frac{499712}{3}L^2\pi^2T^3\zeta_5 - 129024L^2T^3\zeta_7 \\
&+32768L^2\pi^6T^3 - \frac{2828288}{405}L^2\pi^6T^2 - 36864L^2T^2\zeta_3^2 + \frac{11444224}{3}L^2\pi^2T^2\zeta_3 + 20480L^2T^2\zeta_5 \\
&+\frac{2351104}{3}L^2\pi^4T^2 - \frac{7610368}{9}L^2\pi^2T\zeta_3 - 40960L^2T\zeta_5 - \frac{27344896}{45}L^2\pi^4T + 1671168L^2T\zeta_3 \\
&-3817472L^2\pi^2T + \frac{7221248L^2\pi^4}{45} + 2555904L^2\zeta_3 + \frac{17096704L^2\pi^2}{9} - \frac{6914048L^2}{3} + \frac{8192}{9}L\pi^6T^4\zeta_3 \\
&-\frac{133120}{3}L\pi^4T^4\zeta_5 + 369152L\pi^2T^4\zeta_7 - 628992LT^4\zeta_9 + \frac{1176832L\pi^8T^3}{42525} + \frac{210944}{3}L\pi^2T^3\zeta_3^2 \\
&-71680LT^3\zeta_3\zeta_5 + 30720LT^3\zeta_{6,2} + \frac{7872512}{15}L\pi^4T^3\zeta_3 - 1899520L\pi^2T^3\zeta_5 + 867328LT^3\zeta_7 \\
&+\frac{212992}{27}L\pi^6T^3 - \frac{1150976}{15}L\pi^4T^2\zeta_3 + 665600L\pi^2T^2\zeta_5 - 268800LT^2\zeta_7 + \frac{2378752}{405}L\pi^6T^2 \\
&+43008LT^2\zeta_3^2 + \frac{757760}{3}L\pi^2T^2\zeta_3 - 1587200LT^2\zeta_5 - \frac{14838784}{9}L\pi^4T^2 - \frac{2152448L\pi^6T}{2835} \\
&-163840LT\zeta_3^2 + \frac{24051712}{9}L\pi^2T\zeta_3 + 364544LT\zeta_5 + \frac{390412288}{405}L\pi^4T + 2457600LT\zeta_3 \\
&-\frac{39706624}{9}L\pi^2T - \frac{5324800}{9}L\pi^2\zeta_3 + \frac{1998848L\zeta_5}{5} - \frac{34199552L\pi^4}{225} + \frac{9797632L\zeta_3}{3} \\
&+\frac{61534208L\pi^2}{81} - \frac{23560192L}{3} - \frac{11264}{105}\pi^6T^5\zeta_5 + \frac{73216}{5}\pi^4T^5\zeta_7 - 285120\pi^2T^5\zeta_9 \\
&+1271952T^5\zeta_{11} - \frac{10544\pi^{10}T^4}{93555} + \frac{91136}{9}\pi^4T^4\zeta_3^2 - \frac{520832}{3}\pi^2T^4\zeta_3\zeta_5 + 179424T^4\zeta_5^2 \\
&+361088T^4\zeta_3\zeta_7 + \frac{16768}{3}\pi^2T^4\zeta_{6,2} - 26432T^4\zeta_{8,2} + \frac{65536}{45}\pi^6T^4\zeta_3 - 63488\pi^4T^4\zeta_5 \\
&+401408\pi^2T^4\zeta_7 - 508032T^4\zeta_9 + \frac{5137792\pi^6T^3\zeta_3}{2835} - 768T^3\zeta_3^3 + 30976\pi^4T^3\zeta_5 \\
&-\frac{941632}{3}\pi^2T^3\zeta_7 + \frac{2211904T^3\zeta_9}{3} - \frac{142816\pi^8T^3}{14175} + \frac{1183232}{3}\pi^2T^3\zeta_3^2 - 337664T^3\zeta_3\zeta_5 \qquad + \dots
\end{aligned}
$$

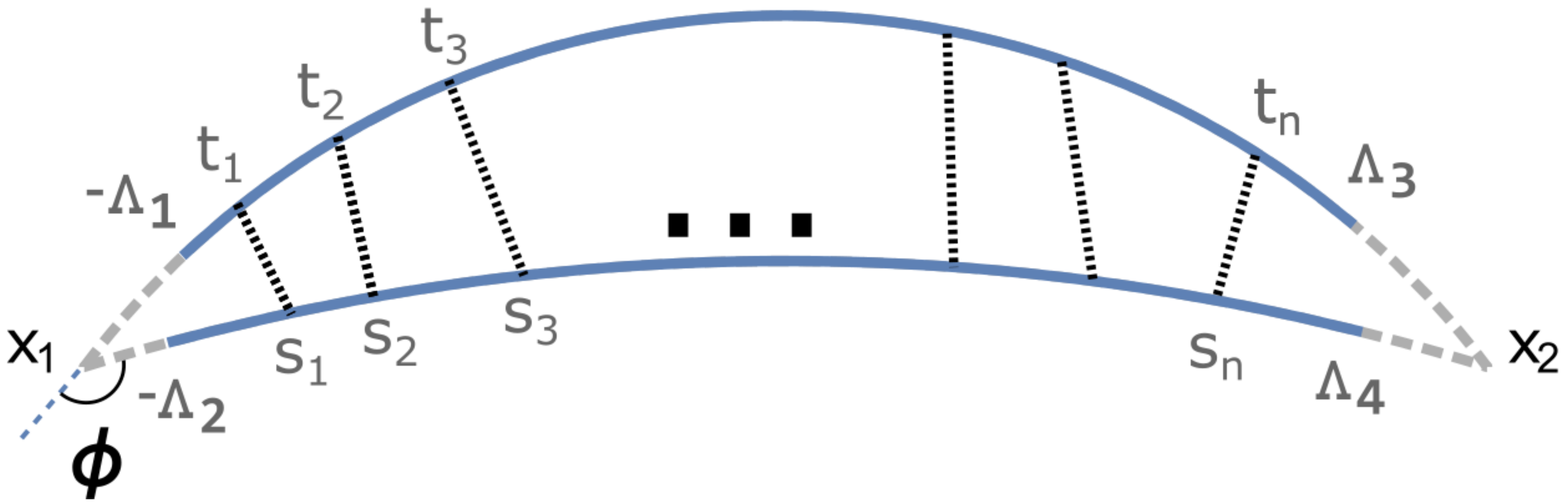
Only Ladder (a.k.a. Fishnet) diagrams survive:



We can hope that 3pt functions is doable in this limit, we can first compute that from diagrams and then try to observe some general structure starting to pop-up from it (if we are lucky)

SET-UP: WHAT WE ARE GOING TO ACTUALLY COMPUTE

$$G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \sum_n$$



propagator

$$\partial_{\Lambda_3}\partial_{\Lambda_4} G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = 2\hat{g}^2 \frac{|\dot{x}(\Lambda_3)||\dot{x}(\Lambda_4)|}{|x(\Lambda_3)-x(\Lambda_4)|^2} G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$$

With the boundary condition

$$G(\Lambda_1, \Lambda_2, -\Lambda_1, \Lambda_4) = G(\Lambda_1, \Lambda_2, \Lambda_3, -\Lambda_2) = 1$$

Nice parametrization:

$$\vec{x} = (\mathrm{Re}(z), \mathrm{Im}(z), 0, 0).$$

$$z_a(s) = z_1 + \frac{(z_1 - z_2)}{e^{-s+id} - 1}, \quad s \in (-\infty, \infty)$$

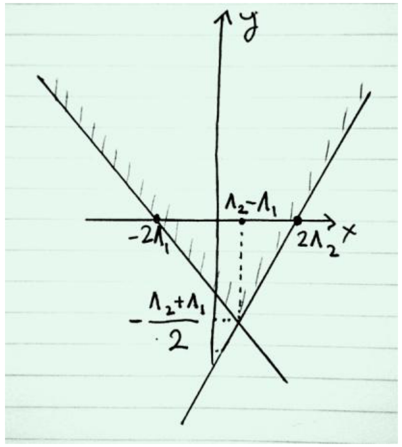
$$z_b(t) = z_1 + \frac{(z_1 - z_2)}{e^{-t+id+i\psi} - 1}, \quad t \in (-\infty, \infty)$$

The propagator became a function of difference:

$$\frac{|\dot{z}_a(s)|\,|\dot{z}_b(t)|}{|z_a(s) - z_b(t)|^2} = \frac{1}{2\,(\cosh(s-t) + \cos\phi)}$$

After the change of coordinates $x = s - t,\ y = \frac{s+t}{2}$ we get a kind-of Schrodinger equation (but quadratic in time):

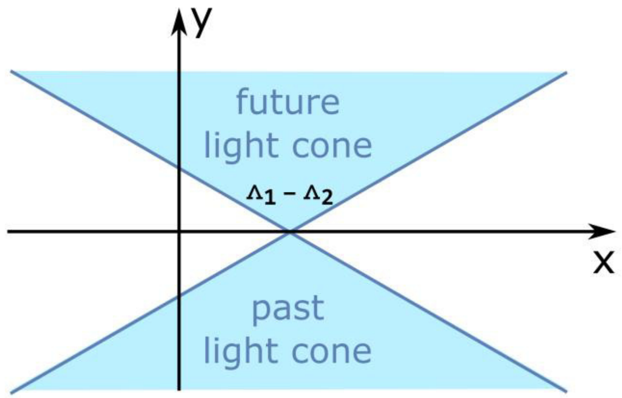
$$\frac{1}{4}\partial_y^2 \tilde{G} = \partial_x^2 \tilde{G} + \frac{2\hat{g}^2}{\cosh x + \cos\phi}\tilde{G}$$



Trick: extend to the whole plane (x,y) by
Reflecting G into the lower light-cone and setting to zero outside the light-cones:

$$G(x,\tilde{y}) = -G(x,-\tilde{y}) \quad , \quad \tilde{y} = y + \frac{\Lambda_1+\Lambda_2}{2}$$

$$\frac{1}{4}\partial_y^2 \tilde{G} = \partial_x^2 \tilde{G} + \frac{2\hat{g}^2}{\cosh x + \cos\phi}\tilde{G}$$

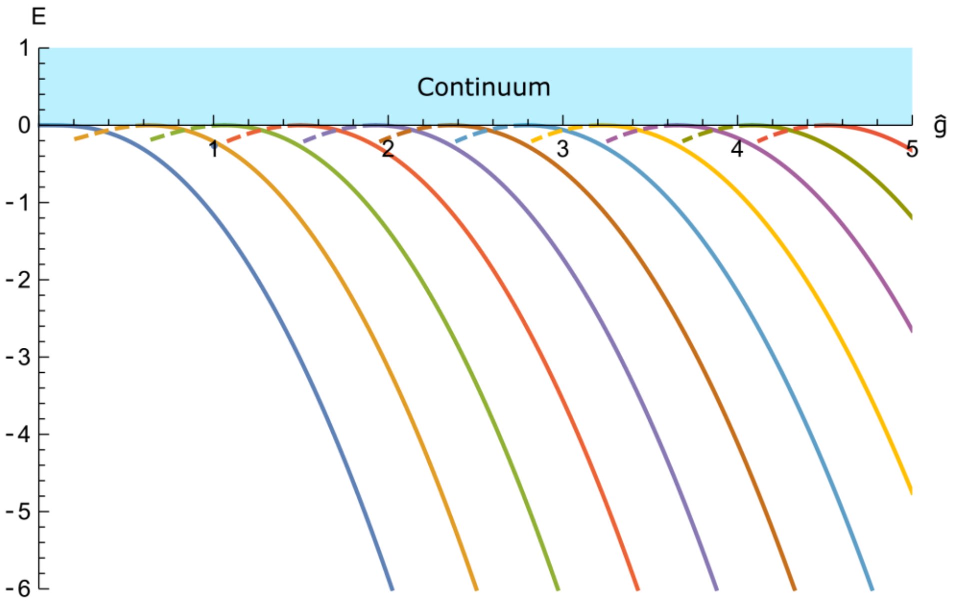


For any fixed y we use the eigenfunctions of the Schrodinger operator:

$$G(x,\tilde{y}) = \sum_n C_n(\tilde{y}) F_n(x)$$

The y-dependent coefficients must satisfy

$$C_n''(\tilde{y}) = -E_n C_n(\tilde{y}) \qquad 4\left[-\partial_x^2 - \frac{2\hat{g}^2}{\cosh x + \cos\phi}\right] F_n(x) = E_n F_n(x)$$



thus

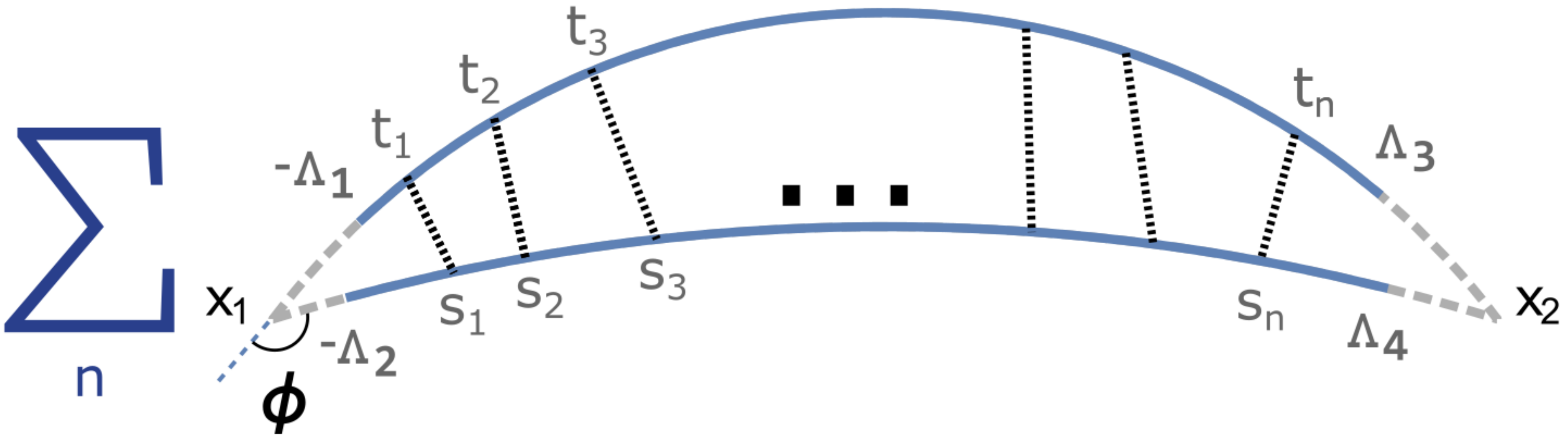
$$G(x,\tilde{y}) = \sum_n c_n \left(e^{+\tilde{y}(-E_n)^{1/2}} - e^{-\tilde{y}(-E_n)^{1/2}} \right) F_n(x)$$

We find the coefficients c_n we notice that

$$G(x,\tilde{y}) = 4\tilde{y}\delta(x - \Lambda_2 + \Lambda_1)$$

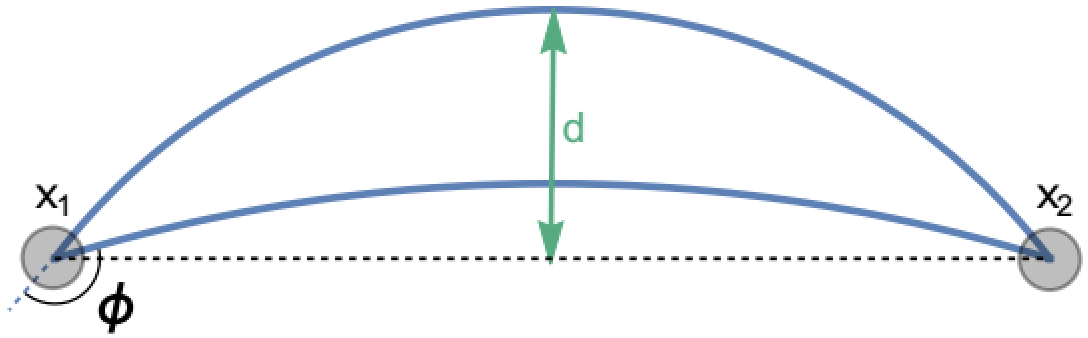
Which fixes the coefficients c_n

$$C_n = \frac{2F_n(\Lambda_2 - \Lambda_1)}{\Omega_n} \qquad \Omega_n = \sqrt{-E_n} > 0$$



$$G(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4)=\sum_n \frac{4F_n(\Lambda_2-\Lambda_1)F_n(\Lambda_3-\Lambda_4)}{\Delta_n}\sinh\left(\Delta_n\frac{\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4}{2}\right)$$

We are ready to compute the 2-point function (including finite part):



$$|z_a(-\Lambda)-z_1|=\epsilon\ \ ,\ \ |z_a(\tilde\Lambda)-z_2|=\epsilon \qquad \Lambda=\tilde\Lambda=\log\left(\frac{x_{12}}{\epsilon}\right)$$

$$G(\Lambda,\Lambda,\Lambda,\Lambda)=\sum_n\frac{4F_n^2(0)}{\Delta_n}\sinh(2\Lambda\Delta_n)\simeq\frac{2F_0^2(0)}{\Delta_0}e^{-2\Delta_0\Lambda}$$

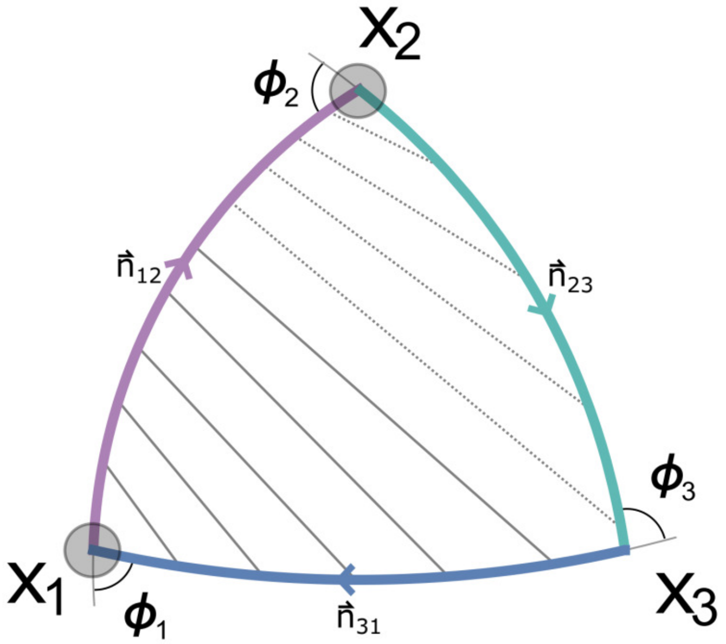
So the result is: $\quad \dfrac{2F_0^2(0)}{\Delta_0}\left(\dfrac{\epsilon}{x_{12}}\right)^{2\Delta_0}$

Next: 3-point correlator:

$$W_{123}^{\bullet\bullet\circ,\,\epsilon} = \frac{\epsilon^{\Delta_1+\Delta_2}}{x_{12}^{\Delta_1+\Delta_2}\, x_{13}^{\Delta_1-\Delta_2}\, x_{23}^{\Delta_2-\Delta_1}} \left(L_{123}\right)^{\Delta_1} \left(L_{231}\right)^{\Delta_2} \left(c_0|_{\Delta_1,\phi_1}\right) \left(c_0|_{\Delta_2,\phi_2}\right) \mathcal{N}_{123}^{\bullet\bullet\circ},$$

Where

$$\mathcal{N}_{123}^{\bullet\bullet\circ} = 2\hat{g}_1^2 \int_{-\infty}^{0} ds \int_{-\infty}^{0} dt\, \frac{F_{\Delta_1,\phi_1}(-\delta x_1 + s - t)\, F_{\Delta_2,\phi_2}(-\delta x_2 - T_{12}(s))\, e^{-\frac{s+t}{2}\Delta_1 - \frac{T_{12}(s)}{2}\Delta_2}}{\cosh(s - t - \delta x_1) + \cos\phi_1}$$



Where

$$e^{T_{12}(s)} = \frac{(1 - e^s)}{1 - e^s \frac{\cos\phi_3 - \cos(\phi_1+\phi_2)}{\cos\phi_3 - \cos(\phi_1-\phi_2)}}.$$

Where

$$L_{123} = \frac{\sqrt{\sin\frac{1}{2}(\phi_1+\phi_2-\phi_3)\,\sin\frac{1}{2}(\phi_1-\phi_2+\phi_3)}}{\sin\phi_1}$$

Where

$$\delta x_1 = \log\frac{\sin\left(\frac{1}{2}(\phi_1-\phi_2+\phi_3)\right)}{\sin\left(\frac{1}{2}(\phi_1+\phi_2-\phi_3)\right)}$$

Where

$$c_n = -\frac{2F_n(0)}{\Delta_n}$$

LET'S TRY WITH INTEGRABILITY

Ladder limit? Only scalars survive
$\Rightarrow$ no-SUSY
$\Rightarrow$ at most we got SO(4,2) left from PSU(2,2|4)
$\Rightarrow$ half of QSC (S^5 part should be trivial)

$$\mathbf{Q}_i^{[+4]} D_0 - \mathbf{Q}_i^{[+2]} \left[D_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} D_0 \right] + \mathbf{Q}_i \left[D_2 - \mathbf{P}_a \mathbf{P}^{a[+2]} D_1 + \mathbf{P}_a \mathbf{P}^{a[+4]} D_0 \right]$$
$$- \mathbf{Q}_i^{[-2]} \left[\bar{D}_1 + \mathbf{P}_a^{[-2]} \mathbf{P}^{a[-4]} \bar{D}_0 \right] + \mathbf{Q}_i^{[-4]} \bar{D}_0 = 0$$

P functions become trivial and the general Baxter equation reduces to

Gromov,FLM 2016

$$\boxed{\left(-2u^2 \cos\phi + 2\Delta u \sin\phi + 4\hat{g}^2\right) q(u) + u^2 q(u-i) + u^2 q(u+i) = 0}$$

$$q_i(u) = \mathbf{Q}_i(u)/\sqrt{u}$$

$$q_1 \sim M_1 e^{\phi u} u^{\Delta}, \quad q_4 \sim M_4 e^{-\phi u} u^{-\Delta}, \quad u \to \infty$$

Quatization condition (comes from analyticity of QSC):

$$\boxed{\Delta = -\frac{2\hat{g}^2}{\sin\phi} \frac{q_1(0)\bar{q}_1'(0) + \bar{q}_1(0) q_1'(0)}{q_1(0)\bar{q}_1(0)}}$$

Relation with the wave functions?

$$F(z) = -i\, e^{-\Delta z/2} \int_{c-i\infty}^{c+i\infty} \frac{q_1(u)}{u}\, e^{w_\phi(z)\, u}\, du, \quad c > 0,$$

Where

$$e^{i w_\phi(z)} = \left(\frac{\cosh \frac{z - i\phi}{2}}{\cosh \frac{z + i\phi}{2}} \right)$$

Kind of a Melline transform. Baxter implies Schrodinger

In terms of q it simplifies drastically!

$$\mathcal{N}_{123}^{\bullet\bullet\circ} = 2\hat{g}_1^2 \int_{-\infty}^{0} ds \int_{-\infty}^{0} dt\, \frac{F_{\Delta_1,\phi_1}(-\delta x_1 + s - t)\, F_{\Delta_2,\phi_2}(-\delta x_2 - T_{12}(s))\, e^{-\frac{s+t}{2}\Delta_1 - \frac{T_{12}(s)}{2}\Delta_2}}{\cosh(s - t - \delta x_1) + \cos\phi_1}$$

$$\mathcal{N}_{123}^{\bullet\bullet\circ} = e^{\frac{\delta_{12}}{2}} \int_{|} du \int_{|} \frac{dv}{v}\, q_{\Delta_1,\phi_1}(u)\, q_{\Delta_2,\phi_2}(v) \left(\frac{e^{(\phi_2 - \phi_3)u - \phi_2 v} - e^{-\phi_1 u + (\phi_1 - \phi_3)v}}{u - v} \right)$$

The structure constant simplifies to:

$$\boxed{C_{123}^{\bullet\bullet\circ} = (K_{123})^{\Delta_1}\, (K_{213})^{\Delta_2}\, \frac{\int_{|} q_1\, q_2\, e^{-\phi_3 u} \frac{du}{2\pi i u}}{\sqrt{\int_{|} q_1\, q_1\, \frac{du}{2\pi i u}}\ \sqrt{\int_{|} q_2\, q_2\, \frac{du}{2\pi i u}}},}$$

$$K_{123} = \frac{\sin \frac{1}{2}(\phi_1 + \phi_2 - \phi_3)}{\sin \phi_1}$$

$$\langle f(u) \rangle \equiv \left(2 \sin \frac{\beta}{2} \right)^{\alpha} \int_{c-i\infty}^{c+i\infty} f(u) \frac{du}{2\pi i u}\ , \quad c > 0$$

Rewriting the Baxter equation in operator form:

$$\hat{O} \equiv \frac{1}{u}\left[(4\hat{g}^2 - 2u^2\cos\phi + 2\Delta u \sin\phi) + u(u-i)D^{-1} + u(u+i)D\right]\frac{1}{u}$$

$$\hat{O}q(u) = 0 \, .$$

We find that the operator is "self-conjugate":

$$\int_{|} q_1(u)\hat{O}q_2(u)du = \int_{|} q_2(u)\hat{O}q_1(u)du$$

Immediate consequences:

- Two solutions with different Δ are orthogonal: $0 = \int_{|} q_1(u)(\hat{O}_1 - \hat{O}_2)q_2(u)du = (\Delta_1 - \Delta_2)2\sin\phi \int_{|} \frac{q_1(u)q_2(u)}{u}du$
- There is a closed equation for the derivative of Δ w.r.t. the coupling: $-\frac{1}{4}\frac{\partial\Delta}{\partial\hat{g}^2} = \frac{\langle q^2 \frac{1}{u}\rangle}{\langle q^2\rangle}$

Can be considered as a correlator of two cusps with Lagrangian, has very similar form to the 3-cusp correlator!

$$\langle f(u)\rangle \equiv \left(2\sin\frac{\beta}{2}\right)^{\alpha} \int_{c-i\infty}^{c+i\infty} f(u)\frac{du}{2\pi i u} \ , \quad c > 0$$

$$\frac{1}{2i\pi}\int_{|} du\,\frac{e^{\beta u}}{u^{\alpha}} = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)}$$

$$q = e^{\phi u} u^{\Delta}\left(1 + \frac{k_1}{u} + \frac{k_2}{u^2} + \dots\right)$$

$$\langle\!\langle q_1 q_2 e^{-\phi_3 u}\rangle\!\rangle = \frac{1}{\Gamma(-\Delta_1 - \Delta_2 + 1)}$$

$$C_{123}^{\bullet\bullet\circ}|_{\phi_i=0} = \frac{\sqrt{\Gamma(1-2\Delta_1)\,\Gamma(1-2\Delta_2)}}{\Gamma(1-\Delta_1-\Delta_2)}$$

[Kim, Kiryu, Komatsy, Nishimura]

$$-\frac{1}{4}\frac{\partial\Delta}{\partial\hat{g}^2} = \frac{\langle\!\langle q^2\,\frac{1}{u}\rangle\!\rangle}{\langle\!\langle q^2\rangle\!\rangle}$$

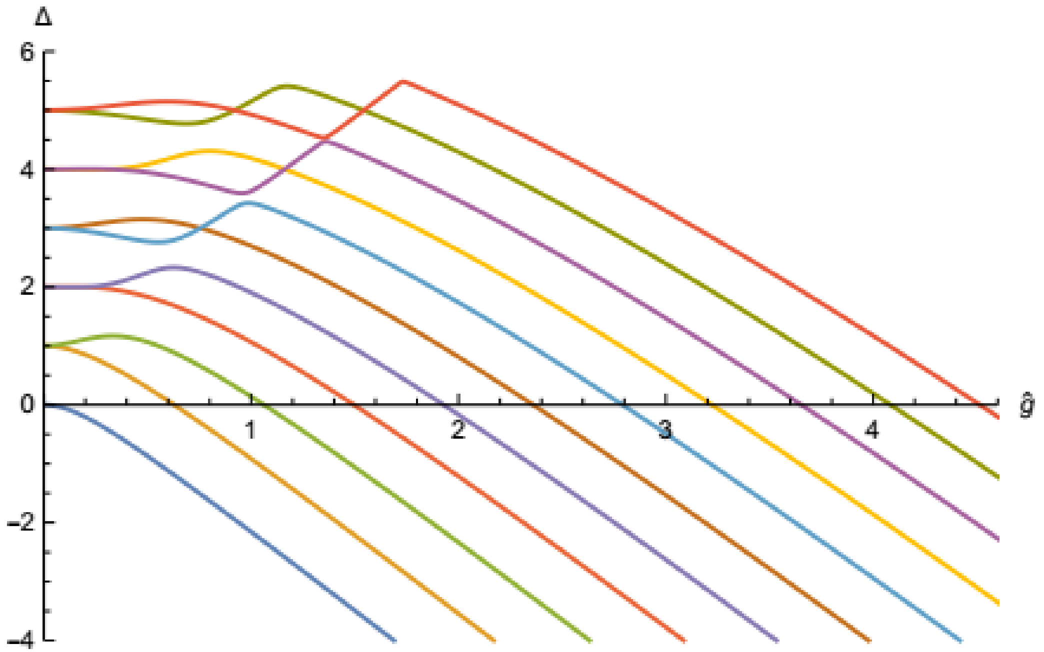
$$\Delta_0 = \frac{1}{2}\left(1 - \sqrt{1 + 16\,\hat{g}^2}\right)$$

[Correa Henn Sever Maldacena]

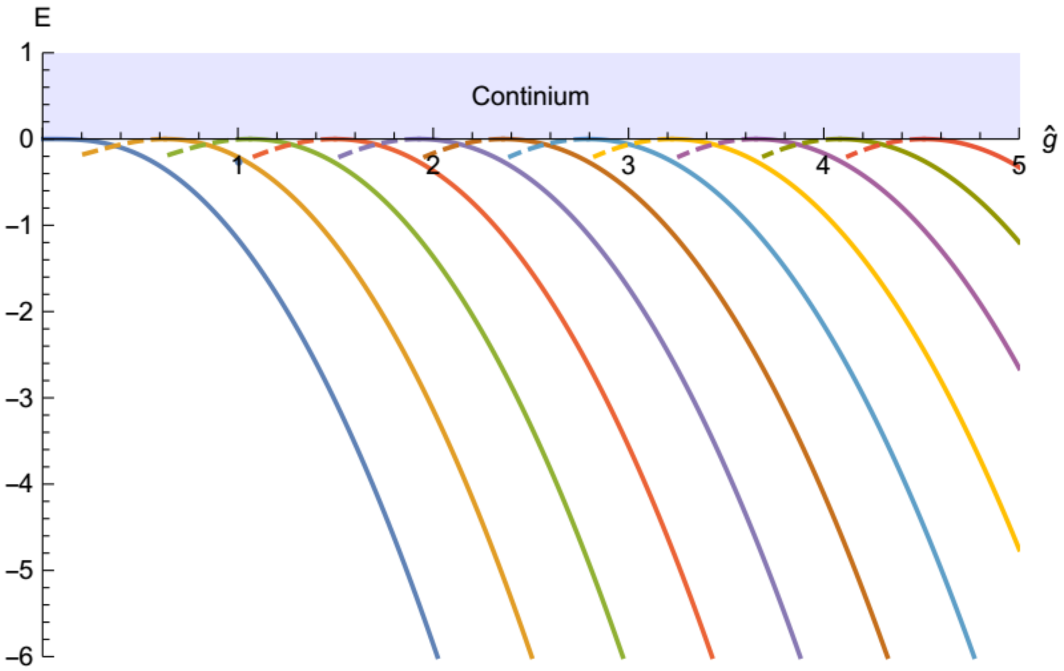
ABOUT EXCITED STATES

The Baxter equation has infinitely many solutions for any coupling:

Baxter:



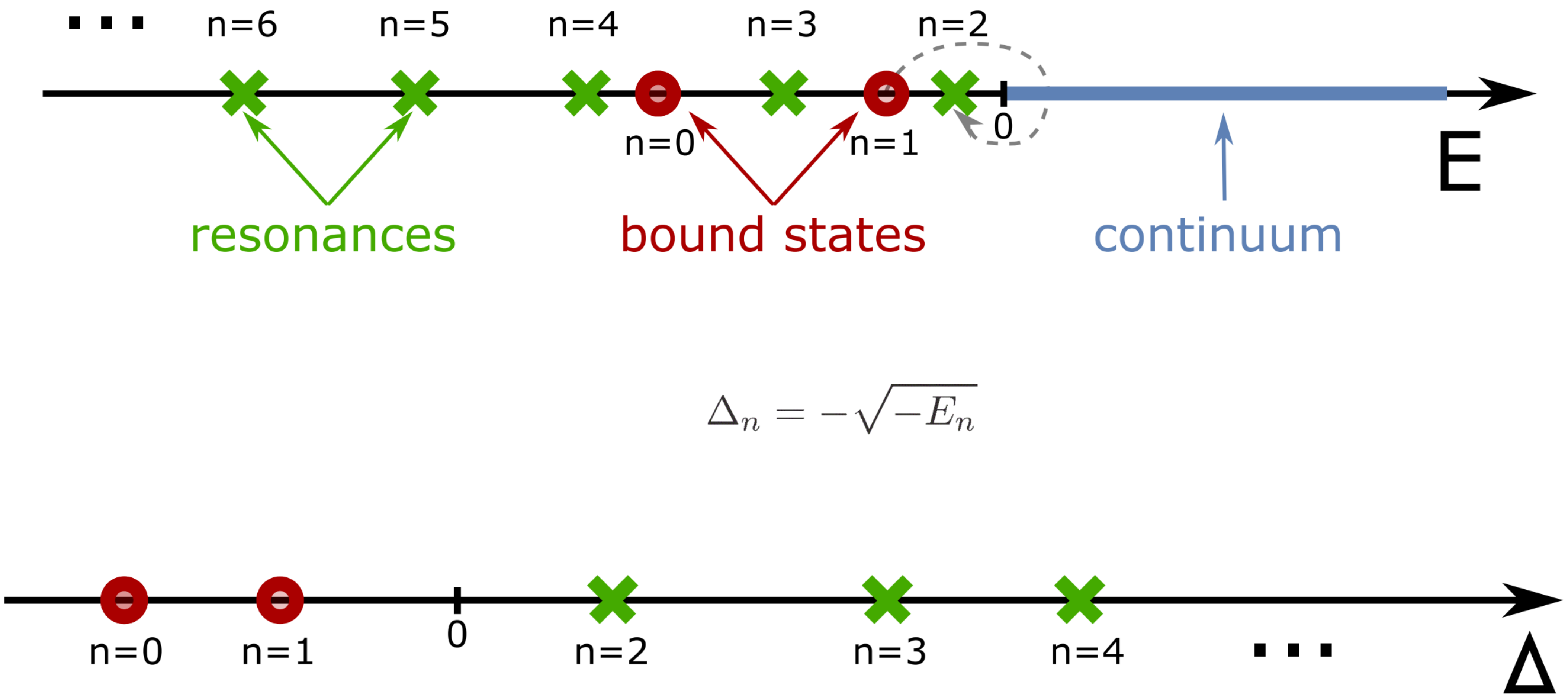
Schrodinger:



$$\Delta_n = -\sqrt{-E_n}$$

Schrodinger has infinitely many resonances! (Spectrum on unphysical sheet of the resolvent, under the cut of the continuous spectrum)

Good luck finding them numerically or analytically from Schrodinger!

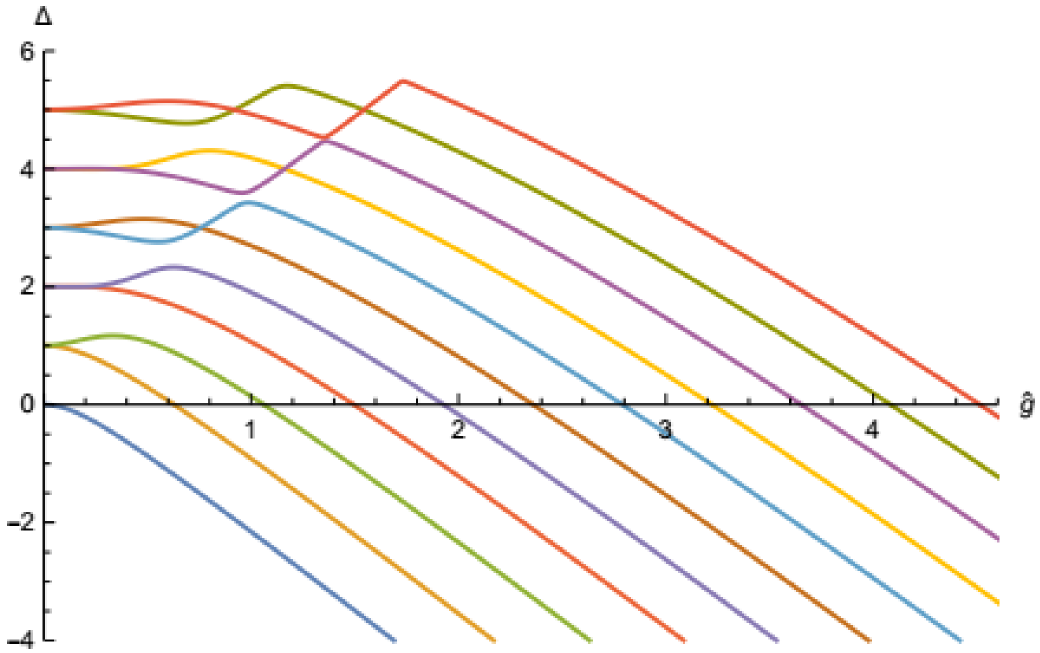

...
n=6
n=5
n=4
n=3
n=2
n=0
n=1
0
E
resonances
bound states
continuum
$\Delta_n = -\sqrt{-E_n}$
n=0
n=1
0
n=2
n=3
n=4
...
Δ

At weak coupling we found:

$$\Delta_{L,\pm} = L \pm \frac{4}{L}\frac{\sin L\phi}{\sin\phi}\hat{g}^2 + \ldots,$$

At strong coupling we found:

$$\frac{\Delta_n}{2\sec\left(\frac{\phi}{2}\right)} = -g + \frac{2n+1}{4} + \frac{(2n^2+2n+1)\cos\phi - 2n^2 - 2n - 3}{64g} + (2n+1)\sin^2\left(\frac{\phi}{2}\right)\frac{\left(n^2+n+1\right)\cos\phi - n^2 - n + 11}{512g^2} + \mathcal{O}\left(g^{-3}\right)$$

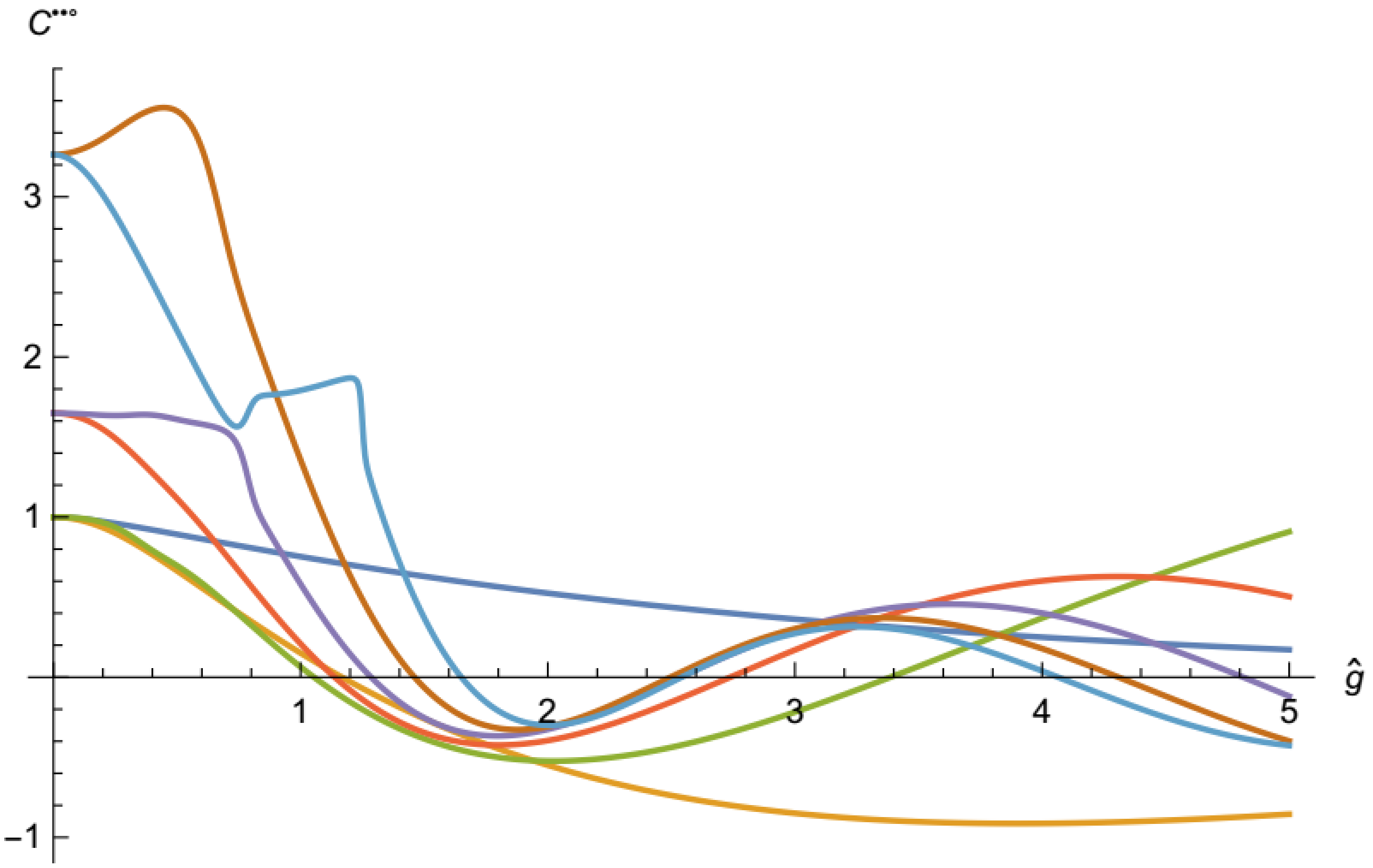


Matches perturbation theory [Henn, Caron-Huot, Brüser] at two loops (was known only for the first excited state)!

$$C_{123}^{\bullet_n \circ\circ} = \frac{\langle\!\langle q_{1,n} e^{\phi_2 u - \phi_3 u} \rangle\!\rangle}{\sqrt{(-1)^n \langle\!\langle q_{1,n}^2 \rangle\!\rangle}}$$

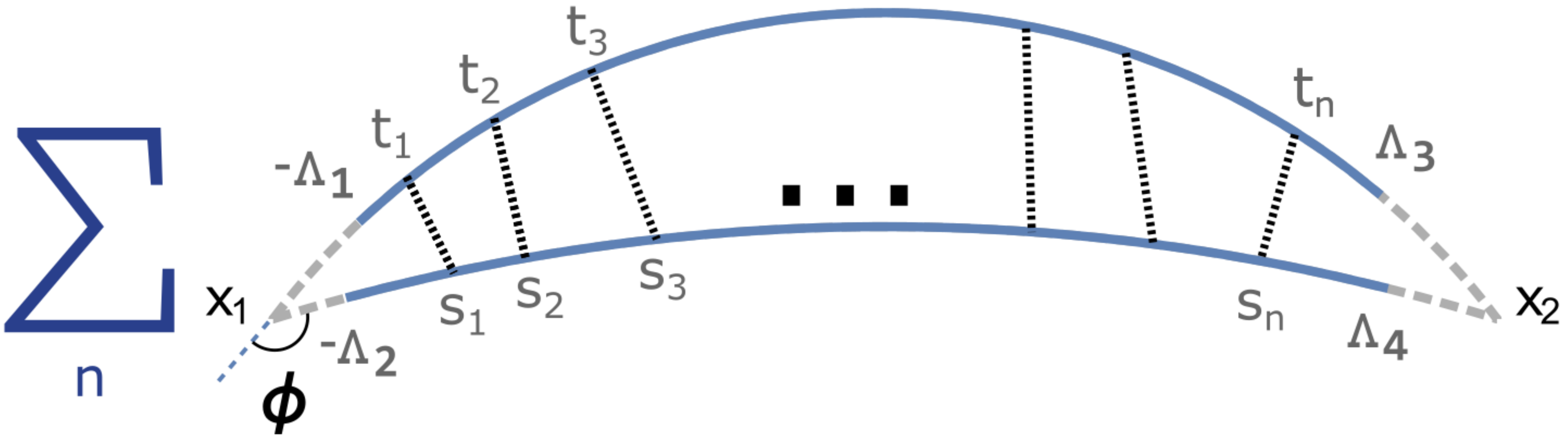
$$C_{123}^{\bullet_n \bullet_m \circ} = \frac{(-1)^m \langle\!\langle q_{1,n} q_{2,m} e^{-\phi_3 u} \rangle\!\rangle}{\sqrt{(-1)^{n+m} \langle\!\langle q_{1,n}^2 \rangle\!\rangle \langle\!\langle q_{2,m}^2 \rangle\!\rangle}}$$

The 3-cusp correlator is given by just the same formula!



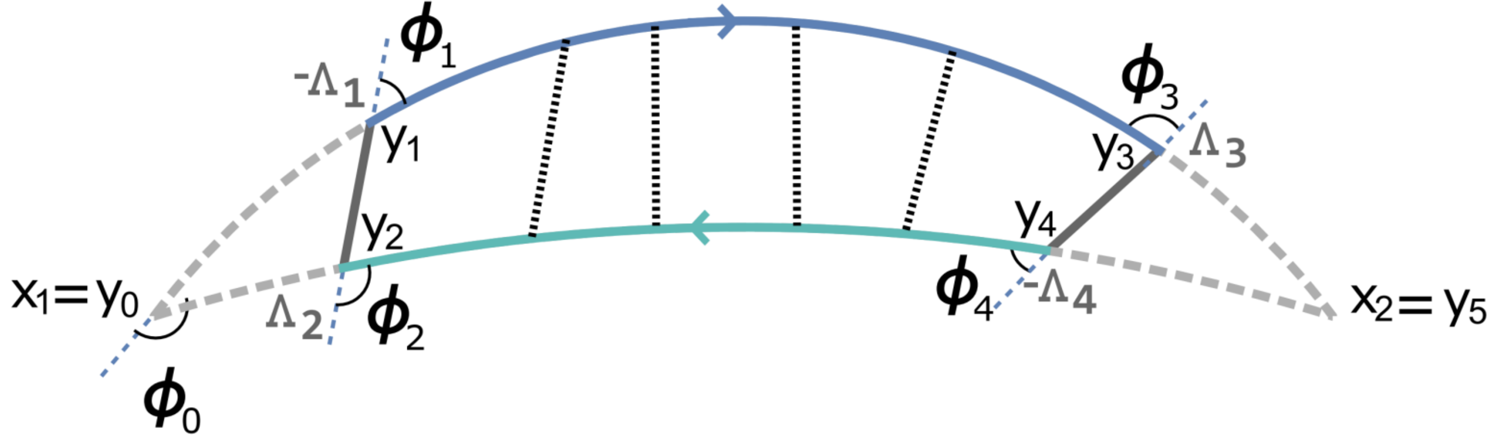
Numerical evaluation faster than for the spectrum!

4POINT AND TWISTED OPE



$$W(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \sum_n \frac{4F_n(\Lambda_2 - \Lambda_1)F_n(\Lambda_3 - \Lambda_4)}{\Delta_n} \sinh\left(\Delta_n \frac{\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4}{2}\right)$$

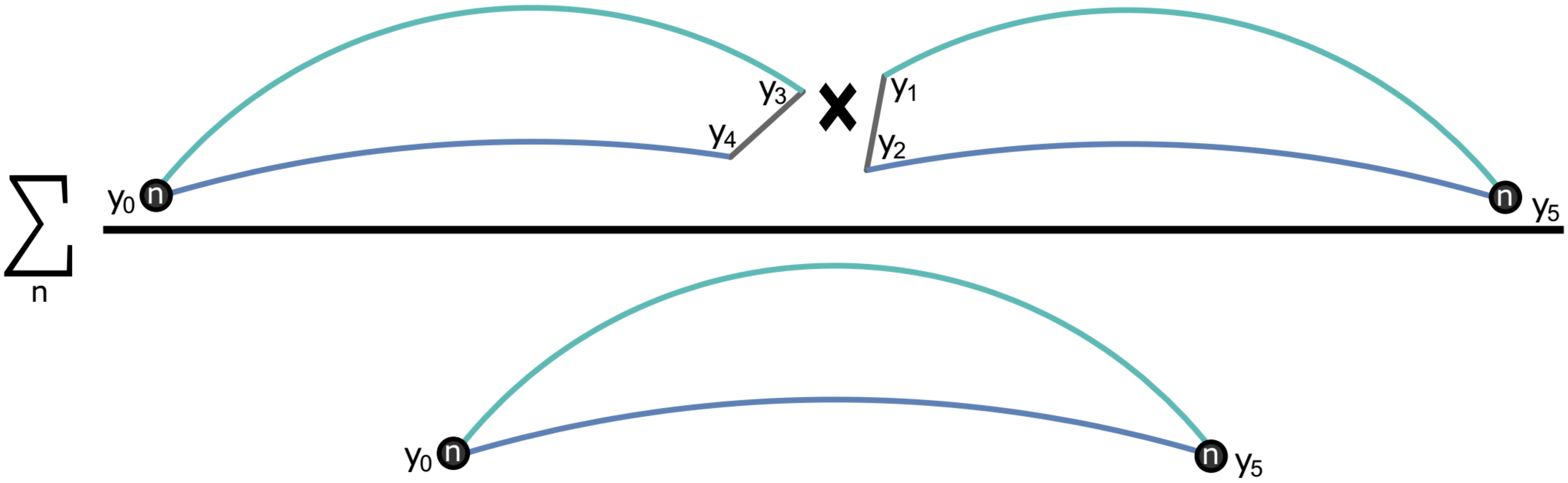
$$C_{123}^{\bullet_n \circ\circ} = \sqrt{\frac{2}{-\Delta_n ||F_n||^2}}\, F_n(\delta x_1)\, (L_{123})^{\Delta_n}$$

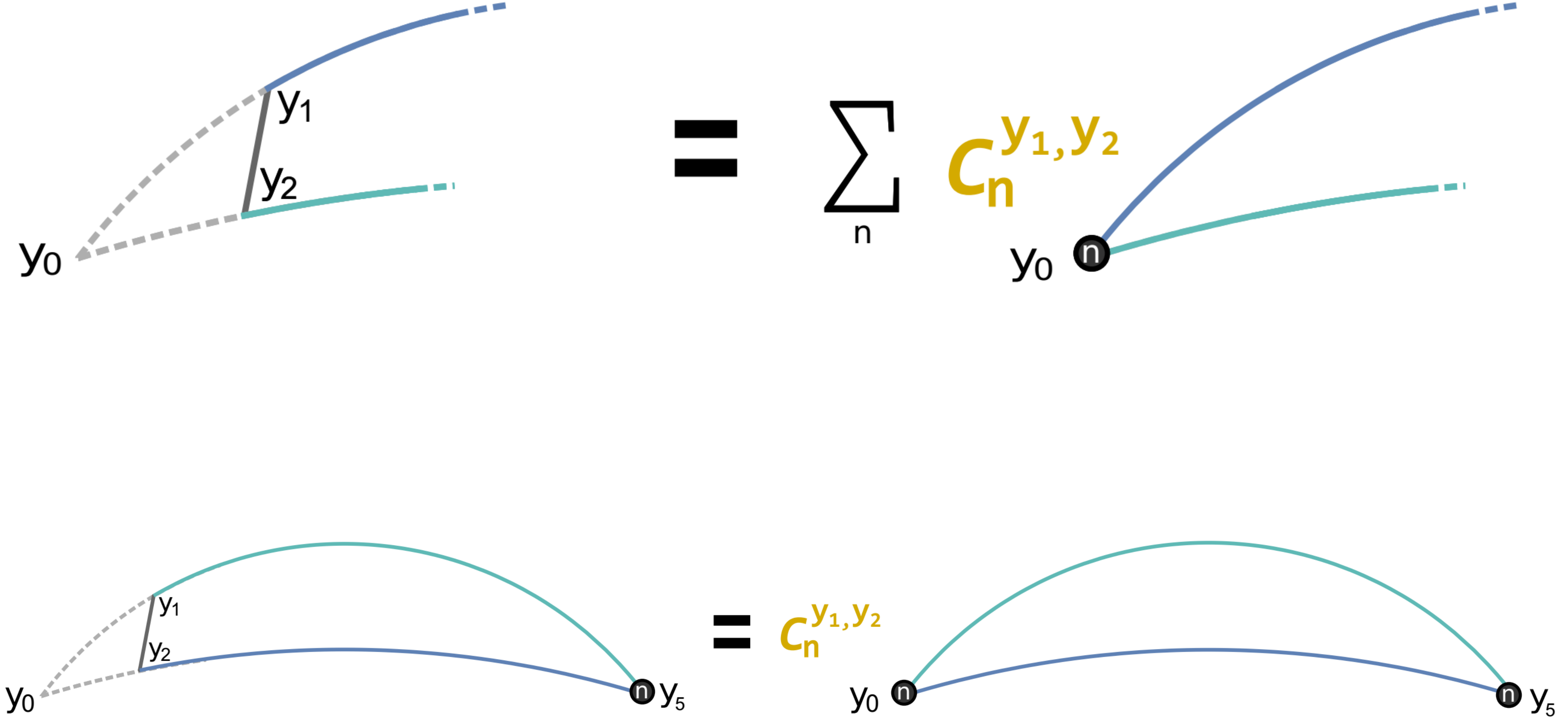


$$G(\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4)=\sum_{n=0}^{\infty}\; C_{012}^{\bullet n\circ\circ}C_{043}^{\bullet n\circ\circ}\left(\frac{e^{-2\Lambda}}{L_{043}L_{012}}\right)^{\Delta_n}$$

$$L_{abc}=\frac{\sqrt{\sin\frac{1}{2}(\phi_a+\phi_b-\phi_c)\,\sin\frac{1}{2}(\phi_a-\phi_b+\phi_c)}}{\sin\phi_a}$$

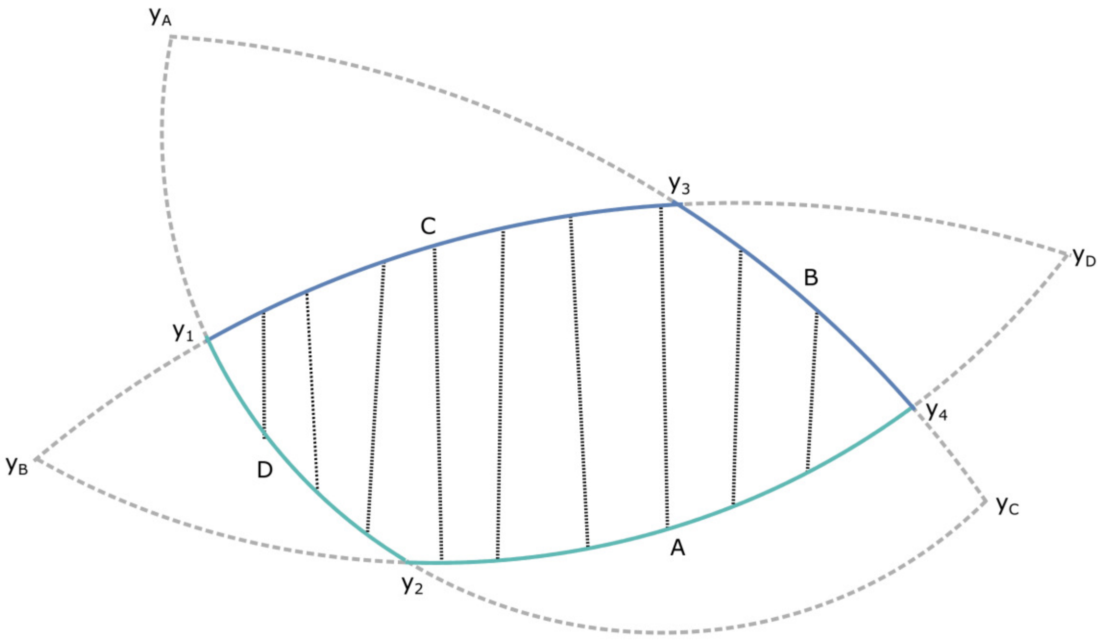
$$\left(\frac{e^{-2\Lambda}}{L_{012}\,L_{034}}\right)^{\Delta_n}=|y_{05}^2|^{\Delta_n}\,\frac{|y_{12}|^{\Delta_n}}{|y_{15}\,y_{25}\,|^{\Delta_n}}\,\frac{|y_{34}|^{\Delta_n}}{|y_{30}\,y_{40}\,|^{\Delta_n}}.$$

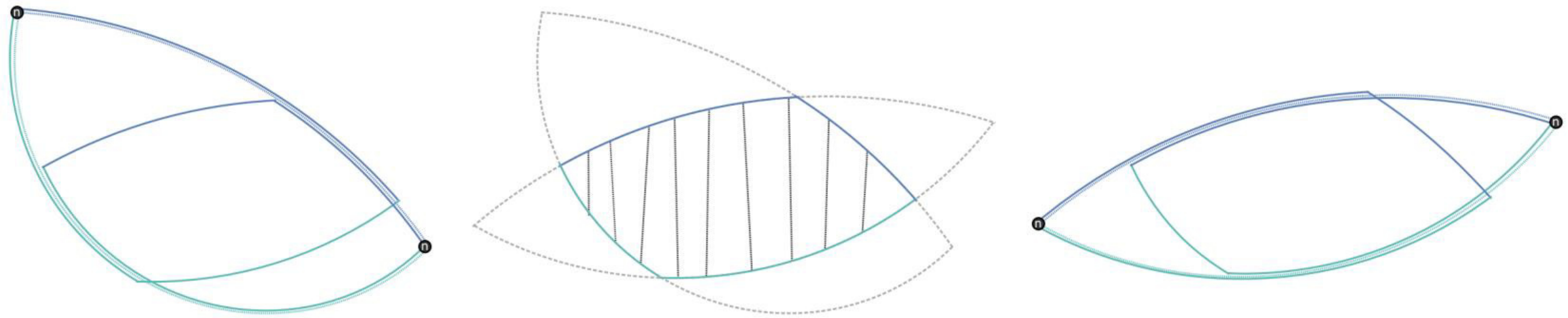

y3
y4
y1
y2
y0
y5
n
y0
y5



y0
y1
y2
=
∑
n
$C_n^{y_1,y_2}$
y0
n
y0
y1
y2
n
y5
=
$C_n^{y_1,y_2}$
y0
n
n
y5

crossing relation?





CONCLUSIONS

- Very simple expression is found but must use q-functions, otherwise complete mess
- Which is good – as QSC gives q-functions for any states, whereas the explicit wave function is not known
- In the SOV (Separation of variables) method q-functions plays the role of the wave functions
  (worked out only for a few simple models)
- Next step – getting 1/cos corrections to the ladder limit and considering more complicated operators at the cusps

After that we solve planar N=4 SYM

4POINT IN FISHNET

N.G. Kazakov, Korchemsky, Negro, Sizov 1706.04167
Grabner, N.G Kazakov, Korchemsky, 1711.0478 *r*
N.G, Kazakov, Korchemsky, *in preparation*

LIMIT

[Kazakov, Gurdagan]

$$\mathcal{L}_{\text{int}} = N_c g \ \text{Tr}\Big[\frac{g}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - g\, e^{-i\epsilon^{ijk}\gamma_k}\phi_i^\dagger\phi_j^\dagger\phi^i\phi^j$$

$$- e^{-\frac{i}{2}\gamma_j^-}\,\bar{\psi}_j\phi^j\bar{\psi}_4 + e^{+\frac{i}{2}\gamma_j^-}\,\bar{\psi}_4\phi^j\bar{\psi}_j + i\epsilon_{ijk}e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\psi^k\phi^i\psi^j$$

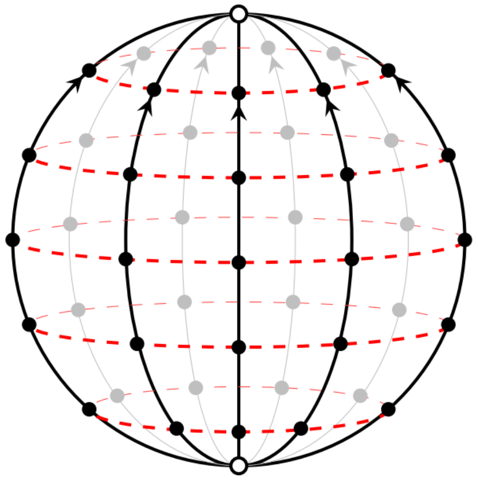
$$- e^{+\frac{i}{2}\gamma_j^-}\,\psi_4\phi_j^\dagger\psi_j + e^{-\frac{i}{2}\gamma_j^-}\,\psi_j\phi_j^\dagger\psi_4 + i\epsilon^{ijk}e^{\frac{i}{2}\epsilon_{jkm}\gamma_m^+}\bar{\psi}_k\phi_i^\dagger\bar{\psi}\Big]\,.$$

$$g \to 0 \qquad g e^{i\gamma} - \text{fixed}$$

$$\mathcal{L}_\phi = \frac{N_c}{2}\text{Tr}\left(\partial^\mu\phi_1^\dagger\partial_\mu\phi^1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi^2 + 2\xi^2\,\phi_1^\dagger\phi_2^\dagger\phi^1\phi^2\right)$$

OBSERVABLE

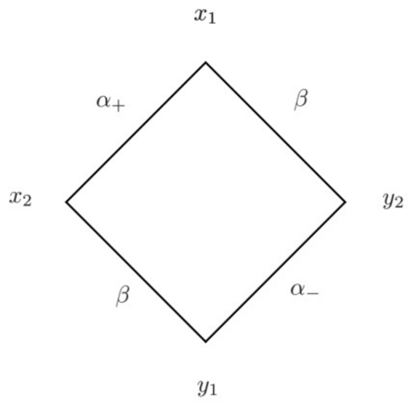
$$\mathcal{O}_{L,n,\ell} = P_{2\ell}(\partial)\mathrm{tr}[\phi_1^L \phi_2^n (\phi_2^\dagger)^n] + \dots ,$$



$$\mathcal{O}_L = \mathrm{Tr}(\phi_1^L) .$$

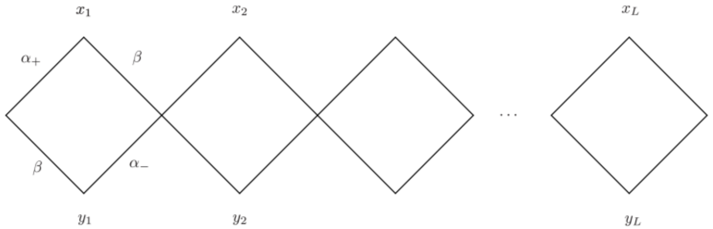
INTEGRABILITY OF FISHNETS

$$R_u(x_1,x_2|y_1,y_2)=\frac{c(u)}{[(x_1-x_2)^2]^{-u-1}[(x_1-y_2)^2(x_2-y_1)^2]^{u+2}[(y_1-y_2)^2]^{-u+1}}\,,$$



Satisfies YB!

$$T_L(u)=\mathrm{tr}_0[R_{10}(u)R_{20}(u)\dots R_{L0}(u)]$$



$$T_{L,u}(\boldsymbol{x}|\boldsymbol{y})=\int d^4x_0d^4y_0d^4z_0\dots d^4w_0\,R_u(x_1,x_0|y_1,y_0)R_u(x_2,y_0|y_2,z_0)\dots R_u(x_L,w_0|y_L,x_0)\,.$$

Too many integrals, but

$$[T_L(u),T_L(v)]=0\,,$$

$$T_{L,u=-1+\epsilon}(\boldsymbol{x}|\boldsymbol{y})=\frac{1}{(16\pi^2\epsilon)^L}\prod_{i=1}^{L}\frac{1}{(x_i-y_i)^2(y_i-y_{i+1})^2}\,,$$

$$\lim_{\epsilon\to0}\frac{\epsilon}{\pi^2[(x-y)^2]^{2-\epsilon}}=\delta^{(4)}(x-y)\,,$$

SO WHAT?

We can diagonalize it using Baxter equation

For L=3 we found

$$\left[\left(\frac{(\Delta-1)(\Delta-3)}{4u^2}+\frac{m}{u^3}-2\right)+D+D^{-1}\right]u^3\left[\left(\frac{(\Delta-1)(\Delta-3)}{4u^2}-\frac{m}{u^3}-2\right)+D+D^{-1}\right]q(u)=0\,.$$

Can write the eigenvalue of H in terms of q

Problem – H itself does not know about coupling and we need

$$\Delta(\xi)$$

OUTPUT FROM QSC – QUANTIZATION CONDITION

q's must satisfy

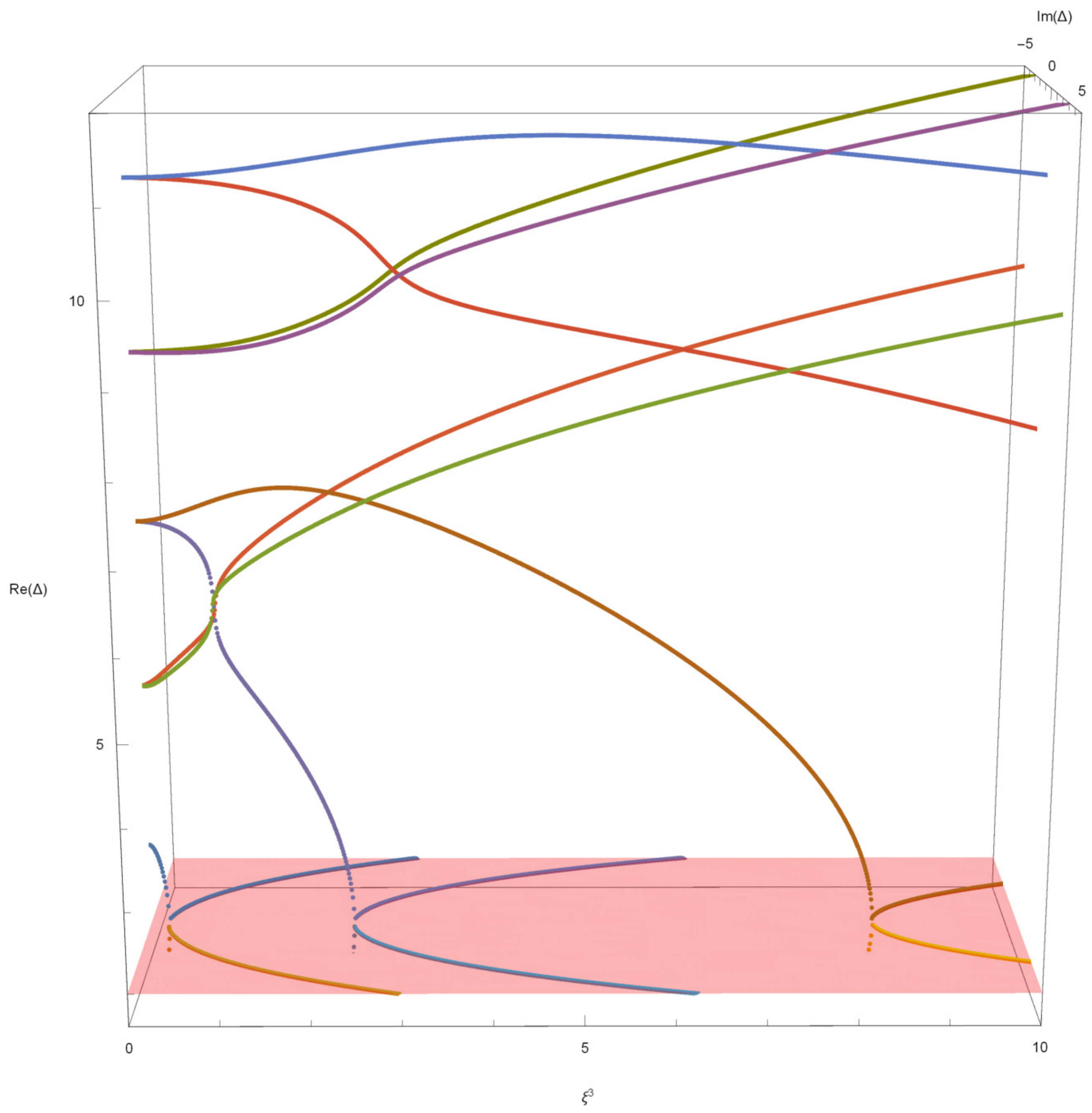
$$q_4(m,0)q_2(-m,0)+q_2(m,0)q_4(-m,0)=0\ .$$

m is nothing but the coupling!

$$m^2=-g^6s^6\ =\xi^6.$$

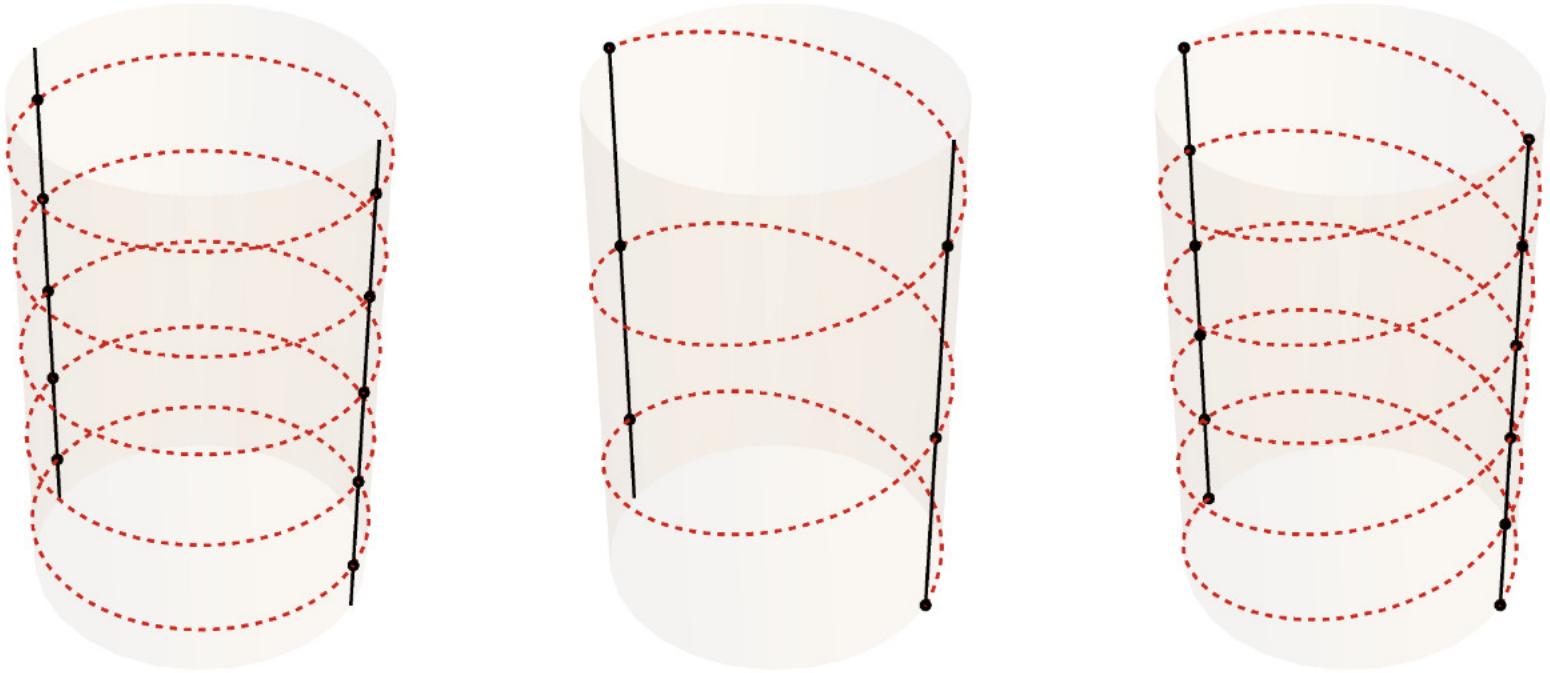
And the "Q-system" boils down to

$$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2}\pm\frac{m}{u^3}-2\right)q(u)+q(u+i)+q(u-i)=0$$


Im(Δ)
−5
0
5
10
Re(Δ)
5
0
5
10
ξ³

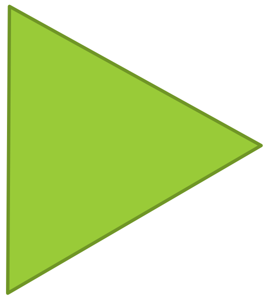
CORRELATION FUNCTIONS?

EXACTLY SOLVABLE 4-POINT CORRELATORS



SOLVING 4-POINT CORRELATOR

$$G_B = 2\left[\mathcal{H}_B + \xi^4 \mathcal{H}_B \star \mathcal{H}_B + \xi^8 \mathcal{H}_B \star \mathcal{H}_B \star \mathcal{H}_B + \ldots\right]$$



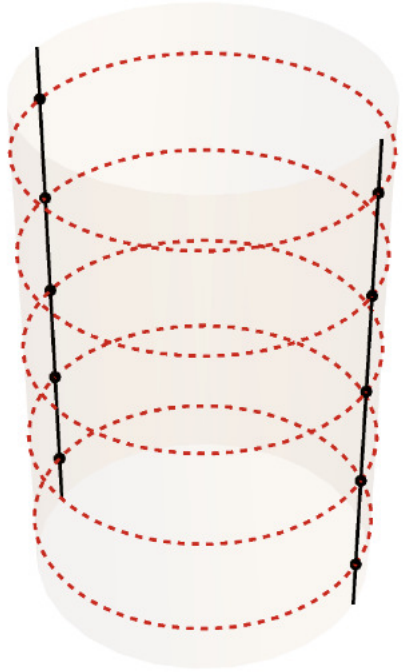
$$\Phi_{\nu,S,x_0}(x_1,x_2) = x_{01}^{-\Delta+\Delta_1-\Delta_2} x_{02}^{-\Delta-\Delta_1+\Delta_2} x_{12}^{\Delta-\Delta_1-\Delta_2} \left(\frac{2x_{01}}{x_{02}x_{12}} (\vec{n}\cdot\vec{x}_{0,2}) - \frac{2x_{02}}{x_{01}x_{12}} (\vec{n}\cdot\vec{x}_{0,1}) \right)^S$$

$$\int d^d x_1 d^d x_2 \Phi_{\nu,S,x_0}(x_1,x_2) H(x_1,x_2|x_3,x_4) = E_{\Delta,s} \Phi_{\nu,S,x_0}(x_3,x_4)$$

$$\mathcal{G}(x_1,x_2|x_3,x_4) = \sum_{S,\Delta} C_{\Delta,S}\, g_{\Delta,S}(u,v)\,,$$

$$C_{\Delta,S} = -2\pi i \left(\frac{1}{4\pi^2}\right)^{-d+\Delta_1+\Delta_2} \mathrm{res}_{\nu=-i/2(\Delta-d/2)} \left(\frac{1}{c_2(\nu,S)} \frac{E^n_{\Delta,S}}{1-\chi E_{\Delta,S}} \right)$$

ALL LOOP CONFORMAL DATA



$$E_{[0]} = \frac{1}{(4\pi^2)^4} \frac{16\pi^4}{(-\Delta + S + 2)(-\Delta + S + 4)(\Delta + S - 2)(\Delta + S)}$$

$$\Delta = 2 + \sqrt{S^2 \pm 2\sqrt{4\xi^4 + (S+1)^2} + 2S + 2}$$

$$C_{\Delta,S} = \frac{c^2(-1)^{-S}\Gamma(S+2)\Gamma(S-\Delta+4)\Gamma\left(\frac{1}{2}(S+\Delta-2)\right)\Gamma\left(\frac{S+\Delta}{2}\right)}{(-\Delta^2+4\Delta+S^2+2S-2)\,\Gamma(S+1)\Gamma\left(\frac{1}{2}(S-\Delta+4)\right)^2\Gamma(S+\Delta-2)}$$

From our exact expression for 4pt function:

Perturbative expansi

$$\mathcal{G}(z,\bar{z})\frac{z-\bar{z}}{z\bar{z}} = g_0 + i\xi^2 g_1 - \xi^4 g_2 - i\xi^6 g_3 + \xi^8 g_4 + i\xi^1 0 g_5 \dots$$

where

$$\begin{aligned}
g_0 &= z - 1 - \frac{1}{z-1} - (z \leftrightarrow \bar{z}) \\
g_1 &= 2H_1\bar{H}_0 + H_1\bar{H}_1 - 2H_2 + 2H_{1,0} - (z \leftrightarrow \bar{z}) \\
-g_2 &= -2\bar{H}_0 H_{1,0} - \bar{H}_0 H_{1,1} - \bar{H}_1 H_{1,0} - 2H_1\bar{H}_{0,0} - H_1\bar{H}_{1,0} + 2H_1\bar{H}_0 + H_1\bar{H}_1 - H_1\bar{H}_2 \\
&\quad +2H_{1,0} + H_{2,1} - 2H_{1,0,0} - H_{1,1,0} - 2H_2 + 2H_3 - (z \leftrightarrow \bar{z}) \\
-g_3 &= -H_1\bar{H}_{1,2} - H_1\bar{H}_{2,0} - 2H_1\bar{H}_{0,0,0} - H_1\bar{H}_{1,0,0} \\
&\quad -2\bar{H}_0 H_{1,2} + 2\bar{H}_0 H_{2,0} - 2\bar{H}_0 H_{1,0,0} - \bar{H}_1 H_{1,2} + \bar{H}_1 H_{2,0} - \bar{H}_1 H_{1,0,0} \\
&\quad -\bar{H}_2 H_{1,0} + 2H_2\bar{H}_{0,0} - 2H_{1,0}\bar{H}_{0,0} + H_2\bar{H}_{1,0} - H_{1,0}\bar{H}_{1,0} + 3H_1\bar{H}_0 \\
&\quad +\frac{3}{2}H_1\bar{H}_1 - H_1\bar{H}_3 - 2H_3\bar{H}_0 - H_3\bar{H}_1 + H_2\bar{H}_2 + 3H_{1,0} \\
&\quad +2H_{2,2} - 2H_{3,0} - 2H_{1,2,0} + 2H_{2,0,0} - 2H_{1,0,0,0} - 8\zeta_3 H_1 - 3H_2 + 2H_4 - (z \leftrightarrow \bar{z}) \\
g_4 &= -H_2\bar{H}_{1,2} - H_2\bar{H}_{2,0} - 2H_2\bar{H}_{0,0,0} - H_2\bar{H}_{1,0,0} + 2\bar{H}_0 H_{1,3} \\
&\quad +\bar{H}_0 H_{3,1} + \bar{H}_0 H_{1,2,0} + \bar{H}_0 H_{1,2,1} - 2\bar{H}_0 H_{2,0,0} - \bar{H}_0 H_{2,1,0} + 2\bar{H}_0 H_{1,0,0,0} \\
&\quad +\bar{H}_1 H_{2,2} + \bar{H}_1 H_{1,2,0} - \bar{H}_1 H_{2,0,0} - \bar{H}_1 H_{2,1,0} + \bar{H}_1 H_{1,0,0,0} \\
&\quad -\bar{H}_2 H_{2,0} + \bar{H}_2 H_{1,0,0} + \bar{H}_3 H_{1,0} + H_{1,2}\bar{H}_{0,0} - 2H_{2,0}\bar{H}_{0,0} - H_{2,1}\bar{H}_{0,0} + 2H_{1,0,0}\bar{H}_{0,0} \\
&\quad +H_{1,2}\bar{H}_{1,0} - H_{2,0}\bar{H}_{1,0} - H_{2,1}\bar{H}_{1,0} + H_{1,0,0}\bar{H}_{1,0} + H_{1,0}\bar{H}_{1,2} + H_{1,0}\bar{H}_{2,0} \\
&\quad +H_1\bar{H}_{2,2} + H_1\bar{H}_{3,0} + 2H_{1,0}\bar{H}_{0,0,0} + H_{1,0}\bar{H}_{1,0,0} + H_1\bar{H}_{1,2,0} + H_1\bar{H}_{2,0,0} + 2H_1\bar{H}_{0,0,0,0} \\
&\quad +H_1\bar{H}_{1,0,0,0} + 8\zeta_3 H_1\bar{H}_0 + 4\zeta_3 H_1\bar{H}_1 - H_2\bar{H}_3 - 6H_1\bar{H}_0 + 2H_4\bar{H}_0 - 3H_1\bar{H}_1 \\
&\quad +H_4\bar{H}_1 + H_1\bar{H}_4 + 8\zeta_3 H_{1,0} - 6H_{1,0} - 2H_{2,3} - H_{3,2} + 2H_{4,0} + 2H_{1,3,0} - H_{2,1,2} + H_{3,1,0} \\
&\quad +H_{1,2,0,0} + H_{1,2,1,0} - 2H_{2,0,0,0} - H_{2,1,0,0} + 2H_{1,0,0,0,0} - 8\zeta_3 H_2 + 6H_2 - 2H_5 - (z \leftrightarrow \bar{z})
\end{aligned}$$

From our exact expression for 4pt function:

Even better sign – boils does further to SVHPL:

$$
\begin{aligned}
g_1 &= 2(L_{10} - L_{01}) \\
g_2 &= -2(L_{10} - L_{01}) \\
&- (2L_{001} + L_{011} - 2L_{100} - L_{110}) \\
g_3 &= -3(L_{10} - L_{01}) \\
&- 2(L_{0001} - L_{0010} + L_{0100} + L_{0101} - L_{1000} - L_{1010} - 4\zeta_3 L_1) \\
g_4 &= -6(L_{10} - L_{01}) \\
&+ (-2L_{00001} + 2L_{00010} - L_{00101} + L_{00110} - 2L_{01000} - 2L_{01001} \\
&\quad -L_{01100} - L_{01101} + 2L_{10000} + 2L_{10010} + L_{10100} + L_{10110} + 8\zeta_3(L_{10} - L_{01})) \\
g_5 &= -\frac{49}{4}(L_{10} - L_{01}) \\
&+ (L_{0001} - L_{0010} + L_{0100} + L_{0101} - L_{1000} - L_{1010} - 4\zeta_3 L_1) \\
&- 2(L_{000001} - L_{000010} + L_{000100} + L_{000101} - L_{001000} - L_{001010} + L_{010000} + L_{010001} \\
&\quad +L_{010100} + L_{010101} - L_{100000} - L_{100010} - L_{101000} - L_{101010} - 12\zeta_5 L_1 - 4\zeta_3 (L_{001} - L_{010} + L_{100} + L_{101}))
\end{aligned}
$$

CONCLUSIONS

Obtained derivation of QSC in this theory from first principles

The theory is perfectly conformal, and the conformal points are integrable!

Lots of data for 3pt and 4pt

SOV for general n-point – ideal settings, very similar structure to cusp