Improving the Existence and Uniqueness Theorems

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There are a variety of directions in which the theorems of the previous chapter can be extended. This chapter is certainly not an exhaustive list, but it will present the main ones. It would be possible to state one grand theorem combining all these extensions together, but the result would be rather unwieldy.

Theorem 1 *Suppose that $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ is open, $F : \Omega \to \mathbf{R}^n$ is continuous and $\partial F/\partial y$ exists and is continuous in Ω. Suppose (ξ, η) is continuous. Then the initial value problem*

$$y'(x) = F(x, y(x)), \quad y(\xi) = \eta$$

has a unique maximally extended solution. In other words there is an open interval I containing ξ and a continuously differentiable function $y : I \to \mathbf{R}^n$ such that $y(\xi) = \eta$, $y'(x) = F(x, y(x))$ for all $x \in I$, and

$$(x, (y(x)) \in \Omega$$

for all $x \in I$, and there is no larger interval to which y can be extended without violating one of these statements. Furthermore, there is only one such interval I and one such function y.

This is not really an extension of the earlier existence and uniqueness theorems. It says both more and less than those theorems. It is more global than the theorems of the preceding chapter. In fact it is as close as one can get to a global existence and uniqueness theorem, but it is also less quantitative. The theorem gives no indication of the size of the interval I.

Before beginning, note that we are trying to solve both forward and backward starting from the initial point ξ, but those problems can be treated separately. It's not necessary to treat them separately, but it will make some things simpler. Furthermore, the forward and backward problems are essentially the same. The reflection $x \to -x$ transforms the backwards problem in the forward problem with ξ replaced by $-\xi$, Ω replaced by the set

$$\{(x, y) \in \mathbf{R} \times \mathbf{R}^n : (-x, y) \in \Omega\}$$

and F replaced by the function which takes (x, y) in the set above to the point $-F(-x, y)$ in $\mathbf{R}^n$. So I will continue to state theorems in terms of solutions defined for x on both sides of ξ, but will give the proofs only for $x \geq \xi$.

Proof: Let T be the set of real numbers $z > \xi$ such that there is a solution to the initial value problem in the interval $[\xi, z]$ and let F be the set of real numbers $z > \xi$ for which there is no such solution. If F is nonempty then, since ξ is a lower bound, it has a greatest lower bound. Call this bound b. For each z in the interval $\xi < z < b$ there is a solution to the initial value problem in the interval $[\xi, z]$. Because of the uniqueness theorem proved in the preceding chapter any two such solutions for different values of z will agree on the part of their domains that they share. We can therefore define $y(x)$ for all x in the range $\xi \leq y < b$ without ambiguity by saying that is the value at x of the solution corresponding to some z in the range $x < z < b$, since we have seen that it doesn't matter which z is chosen. This gives us a solution on the interval $\xi \leq x < b$. This extension cannot be extended, because if it could then b would not be the *greatest* lower bound for F. The case where F is empty is very similar. For each $z > \xi$ there is a solution to the initial value problem in the interval $[\xi, \infty)$. Because of the uniqueness theorem

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any two such solutions for different values of z will agree on the part of their domains that they share, we can unambiguously define, for any $x > \xi$, $y(x)$ be the common value of those solutions on intervals whose domain contains x. In this case it's clear that we can't extend to a larger interval because there is no larger interval. ■

There is an alternate characterisation of the maximally extended solution which is often useful.

Theorem 2 *With notation as in the preceding theorem, let K be a closed bounded subset of Ω. Then there is a $t \in I$ such that*

$$(x, y(x)) \notin K$$

for all $x > t$ in I and an $s \in I$ such that $(x, y(x)) \notin K$ for all $x < s$ in I.

Proof: Since the forward and backward problems are equivalent it is enough to prove the existence of t. If there were no such t then we could find an sequence of $x_n \in I$ with no limit in I such that $(x, y(x_n)) \in K$. By the Heine-Borel Theorem this sequence would have a convergent subsequence. Call its limit (X, Y). Then $(X, Y) \in K$ and hence in $(X, Y) \in \Omega$. By the existentence theorem we can find a local solution with initial conditions $y(X) - Y$. The uniqueness theorem shows that this solution agrees for $x < X$ with the one we already had. The extension to $x > X$ would give us an extension of the maximally extended solution, which is impossible. So the assumption that there is no t is impossible. ■

Differential equation do not always naturally come in the form of a set of equations for each derivative of the unknown. Often we have a general set of n equations for the n unknowns and their derivatives.

Theorem 3 *Suppose that $U \subset \mathbf{R} \times \mathbf{R}^n \times R^n$ be open. Suppose that $\Phi: U \to \mathbf{R}^n$ is continuous in U along with $\partial\Phi/\partial y$ and[1] $\partial\Phi/\partial\upsilon$. Suppose that $(\xi, \eta, \upsilon) \in U$, that $\Phi((\xi, \eta, \upsilon) = 0$, and that $\partial F/\partial\upsilon$ is invertible throughout U. Then there is an open interval I containing ξ and a function $y: I \to \mathbf{R}^n$ such that*

$$\Phi(x, y(x), y'(x)) = 0$$

for all $x \in I$ and

$$y(\xi) = \eta, \quad y'(\xi) = \upsilon.$$

Any two solutions are equal in the intersection of their intervals of definition and there is a unique maximally extended solution.

The hypothesis that $\Phi((\xi, \eta, \upsilon) = 0$ is a necessary condition for the existence of solutions. If there is no solution υ to the equation then there is no y which satisfies the first initial condition $y(\xi) = \eta$ and satisfies the differential equation $\Phi(x, y(x), y'(x)) = 0$ at the point $x = \xi$. The second initial condition is required because there might be more than one υ satisfying $\Phi((\xi, \eta, \upsilon) = 0$.

Proof: By the Implicit Function Theorem there is a ball about (ξ, η) and unique function f continuous in this ball and continuously differentiable with respect to its second argument such that

$$f(\xi, \eta) = \upsilon$$

and

$$\Phi(x, y, f(x, y)) = 0$$

for all (x, y) in the ball. Then y' satisfies the initial value problem above if and only if it satisfies the initial value problem

$$y'(x) = f(x, y(x)), \quad y(\xi) = \eta.$$

So the existence and uniqueness theorems from the preceding chapter give us local existence and uniqueness. We can pass from this to existence and uniqueness of maximally extended solutions as in the first theorem of this chapter. ■

Often differential equations come with parameters. Intuitively we expect that if the equation depends continuously on some parameters then the solution should also depend continuously on those parameters.

Theorem 4 *Suppose that $\Omega \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$ is open and $(\xi, \eta, \zeta) \in \Omega$. Suppose G and $\partial G/\partial G$ are continuous in Ω. Suppose $\{\xi\} \times \{\eta\}\overline{B}_r(\zeta) \subset \Omega$. then there is an open interval I containing ξ and a unique continuous function*

$$y: I \times B_r(\zeta) \to \mathbf{R}^n,$$

[1] We call the final argument of Φ υ.

continuously differentiable in its first argument, such that

$$\frac{\partial y}{\partial x}(x,z) = G(x, y(x,z), z)$$

for all $(x,z) \in I \times B_r(\zeta)$ *and*

$$y(\xi, z) = \eta.$$

It is also possible to allow the initial conditions to depend on the parameters, but this is will not be considered here.

Proof: We extend the system by introducing new variables y_{n+1}, …, y_{n+k}, which satisfy the differential equations $y'_{n+j}(x) = 0$ and the initial conditions $y_{n+j}(\xi) = z_j$. Of course these are satisfied if and only if $y_{n+j}(x) = z_j$ for all $x \in I$. In other words, the given initial value problem with parameters is equivalent to the extended initial value problem

$$y'(x) = F(x, y(x)), \quad y(\xi) = \eta$$

without parameters, where

$$\begin{aligned} &F_j(x, y_1, \ldots, y_n, y_{n+1}, \ldots y_{n+k}) \\ &\quad = G(x, y_1, \ldots, y_n, z_1, \ldots z_k) \end{aligned}$$

for $1 \le j \le n$ and

$$F_{n+j}((x, y_1, \ldots, y_n, y_{n+1}, \ldots y_{n+k}) = 0, \quad \eta_{n+j} = \zeta_j$$

for $1 \le j \le k$. We can therefore apply the theorems we already have, including the theorem about continuous dependence on the initial conditions from the preceding chapter. This works since the parameters of the given problem are initial conditions in the auxiliary problem. ■

A similar idea allows us to convert additional differentiability information about F to additional differentiability of the solution. The simplest case is that of one additional derivative.

Theorem 5 *Suppose* $\Omega \subset \mathbf{R} \times \mathbf{r}^n$ *is open,* $(\xi, \eta) \in \Omega$, *and* $g \colon \Omega \to \mathbf{R}^n$ *is continuously differentiable along with* $\partial g / \partial y$. *The there is an interval* I *containing* ξ *and a unique twice continuously differentible function* $y \colon I \to \mathbf{R}^n$ *satisfying*

$$y'(x) = g(x, y(x)), \quad y(\xi) = \eta.$$

Proof: We introduce additional variables y_{n+1}, …, y_{2n} for the derivatives of y_1, …, y_n. More precisely, we solve the initial value problem

$$y'(x) = f(x, y(x)), \quad y(\xi) = \eta$$

where

$$f_j(x, y_1, \ldots, y_{2n}) = y_{n+j},$$

$$\begin{aligned} &f_{n+j}(x, y_1, \ldots, y_{2n}) \\ &\quad = \frac{\partial g_j}{\partial x}(x, y_1, \ldots, y_n) \\ &\qquad + \sum_{k=1}^{n} \frac{\partial g_j}{\partial y_k}(x, y_1, \ldots, y_n) g_k x, y_1, \ldots, y_n), \end{aligned}$$

$$\eta_{n+j} = g_j(\xi, \eta)$$

for $1 \le j \le n$. ■

One can repeat this procedure for further derivatives. In general we have

Theorem 6 *Suppose* $\Omega \subset \mathbf{R} \times \mathbf{r}^n$ *is open,* $(\xi, \eta) \in \Omega$, *and* $g \colon \Omega \to \mathbf{R}^n$ *is* m *times continuously differentiable along with* $\partial g / \partial y$. *The there is an interval* I *containing* ξ *and a unique* $m+1$ *times continuously differentible function* $y \colon I \to \mathbf{R}^n$ *satisfying*

$$y'(x) = g(x, y(x)), \quad y(\xi) = \eta.$$