Existence and Uniqueness Theorems

John Stalker[*]

March 3, 2015

In these notes, $B_r(\xi)$ denotes the open ball of radius r about ξ and $\overline{B}_r(\xi)$ is the closed ball of radius r about ξ. This notation will be used in Euclidean spaces of any dimension, including dimension 1.

Theorem 1 *Suppose that $\xi \in \mathbf{R}$, $\eta \in \mathbf{R}^n$, $r > 0$ and $s > 0$. Suppose*

$$F \colon \overline{B}_r(\xi) \times \overline{B}_s(\eta) \to \mathbf{R}^n$$

satisfies

$$\left\| \frac{\partial F}{\partial y}(x, y) \right\| \le M$$

for each $x \in \overline{B}_r(\xi)$ and $y \in \overline{B}_s(\eta)$, where the $\partial F/\partial y$ is a matrix of partial derivatives and the norm is the usual matrix norm. Then

$$\|F(x, y_a) - F(x, y_b)\| \le M \|y_a - y_b\|$$

for all $x \in \overline{B}_r(\xi)$ and $y_a, y_b \in \overline{B}_s(\eta)$.

Before beginning the proof, note that the norm is continuous and $\overline{B}_r(\xi) \times \overline{B}_s(\eta)$ is closed and bounded, so if $\partial F/\partial y$ is continuous then there must be such an M.

Proof: Let

$$u(t, y_a, y_b) = t y_a + (1 - t) y_b.$$

and

$$f(t, x, y_a, y_b) = F(x, u(t, y_a, y_b)).$$

Then

$$F(x, y_a) - F(x, y_b) = f(1, x, y_a, y_b) - f(0, x, y_a, y_b).$$

By the Fundamental Theorem of the Calculus,

$$f(1, x, y_a, y_b) - f(0, x, y_a, y_b) = \int_0^1 \frac{\partial f}{\partial t}(t, x, y_a, y_b)\, dt.$$

By the Chain Rule,

$$\frac{\partial f}{\partial t} u(t, x, y_a, y_b) = \frac{\partial F}{\partial y}(x, u(t, y_a, y_b)) \frac{\partial u}{\partial t}(t, y_a, y_b).$$

From the definition of u we see that

$$\frac{\partial u}{\partial t}(t, y_a, y_b) = (y_b - y_a).$$

Since the matrix norm is submultiplicative

$$\left\| \frac{\partial f}{\partial t} u(t, x, y_a, y_b) \right\| \le M \|y_a - y_b\|.$$

Integrating this inequality gives

$$\|F(x, y_a) - F(x, y_b)\| \le M \|y_a - y_b\|$$

as required. ■

From now on we assume that

$$F \colon \overline{B}_r(\xi) \times \overline{B}_s(\eta) \to \mathbf{R}^n$$

satisfies

$$\|F(x, y_a) - F(x, y_b)\| \le M \|y_a - y_b\|$$

for all $x \in \overline{B}_r(\xi)$ and $y_a, y_b \in \overline{B}_s(\eta)$ and that

$$\|F(x, \eta)\| \le N$$

for all $x \in \overline{B}_r(\xi)$. If F is continuous then there is such an N.

[*]School of Mathematics, TCD,

Theorem 2 *Suppose $0 < \sigma < s$. There is a $\rho > 0$ such that if $y_0 \in B_\sigma(\eta)$ and $y\colon B_\rho(\xi) \to \mathbf{R}^n$ satisfies*

$$y'(x) = F(x, y(x))$$

for $x \in B_\rho(\xi)$ and

$$y(\xi) = y_0$$

then

$$y(x) \in B_s(\eta)$$

for all $x \in B_\rho(\xi)$.

Proof: It will be shown that this holds with

$$\rho = \min\left(r, \frac{1}{M}\log\left(\frac{Ms+N}{M\sigma+N}\right)\right).$$

By continuity there is a $\rho' > 0$ such that

$$y(x) \in B_s(\eta)$$

for all $x \in B_{\rho'}(\xi)$. Suppose that $x \in B_{\rho'}(\xi)$. The derivative of the the norm of a matrix valued function is bounded by the norm of the derivative. We apply this to

$$z(x) = y(x) - \eta,$$

obtaining

$$\left|\frac{d}{dx}\|z(x)\|\right| \leq \|z'(x)\| = \|y'(x)\| = \|F(x, y(x))\|.$$

By the triangle inequality

$$\|F(x, y(x))\| \leq \|F(x, y(x)) - F(x, \eta)\| + \|F(x, \eta)\|.$$

By hypothesis

$$\|F(x, y(x)) - F(x, \eta)\| \leq M\|y(x) - \eta\| = M\|z(x)\|$$

and

$$\|F(x, \eta)\| \leq N.$$

Thus

$$\left|\frac{d}{dx}\|z(x)\|\right| \leq M\|z(x)\| + N.$$

By Gronwall's inequality

$$\|z(x)\| \leq \left(\|z(\xi)\| + \frac{N}{M}\right)\exp(M|x-\xi|) - \frac{N}{M}.$$

Now

$$\|z(\xi)\| = \|y(\xi) - \eta\| = \|y_0 - \eta\| < \sigma$$

and

$$|x - \xi| < \rho',$$

so

$$\|z(x)\| \leq \left(\sigma + \frac{N}{M}\right)\exp(M\rho') - \frac{N}{M} < s.$$

The Bootstrap Lemma from the preceding chapter then allows us to conclude that

$$\|z(x)\| < s$$

for all $x \in B_\rho(\xi)$. ■

Theorem 3 *If y_a and y_b satisfy the differential equation*

$$y'(x) = F(x, y(x))$$

for all $x \in B_\rho(\xi)$ with initial conditions

$$y_a(\xi) = z_a, \quad y_b(\xi) = z_b$$

then

$$\|y_a(x) - y_b(x)\| \leq \|z_a - z_b\| \exp(M|x - \xi|).$$

Proof: If y_a and y_b are solutions then

$$\begin{aligned}\left|\frac{d}{dx}\|y_a(x) - y_b(x)\|\right| &\leq \|y_a'(x) - y_b'(x)\| \\ &= \|F(x, y_a(x)) - F(x, y_b(x))\| \\ &\leq M\|y_a(x) - y_b(x)\|.\end{aligned}$$

The last step is justified because, by the preceding theorem,

$$y_a(x), y_b(x) \in B_s(\eta)$$

for all $x \in B_\rho(\xi)$. Applying Gronwall again,

$$\|y_a(x) - y_b(x)\| \leq \|y_a(\xi) - y_b(\xi)\| \exp(M|x - \xi|).$$

■

There are two important corollaries.

Theorem 4 *The initial value problem*

$$y'(x) = F(x, y(x)), \quad y(\xi) = y_0$$

has at most one solution in the interval $B_\rho(\xi)$.

Assume there are two solutions, y_a and y_b. Then the preceding theorem, with $z_a = z_b = y_0$ gives **Proof:**

$$\|y_a(x) - y_b(x)\| = 0$$

and hence $y_a = y_b$. ∎

Theorem 5 *Suppose that for each $z \in B_\sigma(x)$ there is a solution to the initial value problem*

$$y'(x) = F(x, y(x)), \quad y(\xi) = y_0$$

in $B_\rho(\xi)$. This solution depends continuously on z, uniformly in x.

Proof: Suppose $\epsilon > 0$. Let $\delta = \epsilon \exp(-M\rho)$. Then if

$$\|z_a - z_b\| < \delta$$

and y_a and y_b are the solutions with those initial conditions then, by the theorem,

$$\|y_a(x) - y_b(x)\| \leq \delta \exp(M|x - \xi|) < \epsilon$$

for all $x \in B_\rho(\xi)$. This is the definition of continuity. It is uniform because the δ we found depends only on ϵ, not on x. ∎

To make the preceding theorem non-trivial we still need to show that solutions exist.

Theorem 6 *For any $z \in B_\sigma(\eta)$ there is a continously differentiable $y\colon B_\rho(\xi)$ such that*

$$y'(x) = F(x, y(x)), \quad y(\xi) = z.$$

Proof: We define, inductively,

$$y_0(x) = \eta, \quad y_{n+1}(x) = z + \int_{x_0}^{x} F(t, y_n(t))\, dt.$$

$$\begin{aligned} y_1(x) - y_0(x) &= y_1(\xi) - y_0(\xi) + \int_{x_0}^{x} (y_1'(t) - y_0'(t))\ dt \\ &= z - \eta + \int_{x_0}^{x} F(t, y_0(t))\, dt. \\ &= z - \eta + \int_{x_0}^{x} F(t, \eta)\, dt. \end{aligned}$$

Now

$$\|z - \eta\| < \sigma$$

and

$$\|F(t, \eta)\| \leq N$$

so

$$\|y_1(x) - y_0(x)\| \leq \sigma + N|x - x_0|.$$

By generalised induction we show that for all $n \geq 1$ and all $x \in B_\rho(\xi)$

- $$\|y_n(x) - y_{n-1}(x)\| \leq g_n(x)$$

  where

  $$g_n(x) = \frac{\sigma M^{n-1}|x - x_0|^{n-1}}{(n-1)!} + \frac{M^{n-1}N|x - x_0|^n}{n!},$$

- $$\|y_n(x) - \eta\| \leq \sum_{m=1}^{n} g_m(x).$$

In case $n = 1$ both of these are just the statement

$$\|y_1(x) - y_0(x)\| \leq \sigma + N|x - x_0|.$$

proved above. For the induction step we assume the two inequalities above and the corresponding inequalities with n replaced by any smaller integer. Using these, we prove the corresponding inequalities with n replaced by $n+1$. First of all, $g_m(x)$ is non-negative, so

$$\begin{aligned} \sum_{m=1}^{n} g_m(x) &\leq \sum_{m=1}^{\infty} g_m(x) \\ &= \left(\sigma + \frac{N}{M}\right) \exp(M|x - x_0|) - \frac{N}{M} \\ &< s. \end{aligned}$$

Thus

$$y_n(x) \in B_s(\eta)$$

for all $x \in B_\rho(\xi)$. Since we also have the corresponding inequality with n replaced by $n - 1$. From this it follows that

$$\|F(x, y_n(x)) - F(x, y_{n-1}(x)\| \leq M\|y_n(x) - y_{n-1}(x)$$

for all $x \in B_\rho(\xi)$. By the other part of the induction hypothesis the right hand side is bounded by $Mg_n(x)$. Since

$$y_{n+1}(x) - y_n(x) = \int_{x_0}^{x} \left(F(x, y_n(x)) - F(x, y_{n-1}(x)\right) \, dx$$

we see that

$$\|y_{n+1}(x) - y_n(x)\| \leq \int_{x_0}^{x} M g_n(t) \, dt = g_{n+1}(x).$$

That's half of what we needed to prove for the induction step. The other half follows because, by the triangle inequality,

$$\|y_{n+1}(x) - \eta\| = \|y_{n+1}(x) - y_n(x)\| + \|y_n(x) - \eta\| .$$

For the first norm on the right we use the inequality just obtained and for the second we use the induction hypothesis. The result is

$$y_{n+1}(x) - y_n(x) \leq g_{n+1}(x) + \sum_{m=1}^{n} g_m(x) = \sum_{m=1}^{n+1} g_m(x).$$

This is the other half of what we needed to prove, thus completing the induction. ■