MA 2326
Assignment 7
Due never

`Id: 2326-s2015-7.m4,v 1.1 2015/04/17 21:39:06 john Exp john`

1. A class of popular models for the dynamics of the populations of two species is

$$x'(t) = ax(t)^2 + bx(t)y(t) + cx(t), \quad y'(t) = dx(t)y(t) + ey(t)^2 + fy(t).$$

   (a) What are the equilibria?
   Solution: To find the equilibria we need to solve the equations

$$ax^2 + bxy + cx = 0, \quad dxy + ey^2 + fy = 0.$$

These factor as

$$x(ax + by + c) = 0, \quad y(dx + ey + f) = 0.$$

The first equation tells us that $x = 0$ or $ax + by + c = 0$. The second tells us that $y = 0$ or $dx + ey + f = 0$. So in total there are four possibilities.

$$x = 0, \quad y = 0,$$

or

$$x = 0, \quad dx + ey + f = 0,$$

or

$$ax + by + c = 0, \quad y = 0,$$

or

$$ax + by + c = 0, \quad dx + ey + f = 0.$$

The first three are trivial to solve and the last is only slightly harder. We thus find the four equilibria

$$(0,0), \quad (0,-f/e), \quad (-c/a,0), \quad \left(\frac{bf-ce}{ae-bd}, \frac{cd-af}{ae-bd}\right).$$

Depending on the parameters, not all of these need exist, as the denominators can be zero. Also, they needn't all be distinct. Those that do exist may not be biologically meaningful, as they may give negative populations for one or both species.

(b) Find the linearisation about each equilibrium.
The linearisation about (ξ,η) is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x-\xi \\ y-\eta \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2a\xi + b\eta + c & b\xi \\ d\eta & d\xi + 2e\eta + c \end{pmatrix}.$$

Substituting the four equilibria we found above gives

$$A = \begin{pmatrix} c & 0 \\ 0 & f \end{pmatrix}$$

or

$$A = \begin{pmatrix} c - bf/e & 0 \\ -df/e & -f \end{pmatrix}$$

or

$$A = \begin{pmatrix} -c & -bc/a \\ 0 & f - cd/a \end{pmatrix}$$

or

$$A = \frac{1}{ae-bd} \begin{pmatrix} abf - ace & b^2 f - bce \\ -adf + cd^2 & -aef + cde \end{pmatrix}.$$

(c) Determine, where possible, the stability properties of each equilibrium.
Solution: The first equilibrium is strictly stable if both c and f are negative and is unstable if at least one is positive. The second equilibrium is stable if f and $bf - ce$ are both negative and unstable if either is positive. The third equilibrium is stable if c and $cd - af$ are both negative and unstable if either is positive. The fourth equilibrium is more complicated. We compute the trace and determinant of A, obtaining

$$\frac{abf - ace - aef + cde}{ae - bd}$$

and
$$\frac{-abf^2 + acef + bcdf - c^2de}{ae - bd}.$$
If the trace is negative and the determinant positive then both roots of the characteristic polynomial have negative real parts, so the equilibrium is stable. If the trace is positive or the determinant is negative then there is at least one root with positive real part and so the equilibrium is unstable.

2. The time reversed Van der Pol equation is, after reduction of order,
$$x'(t) = y(t), \quad y'(t) = -x(t) + \mu(x(t)^2 - 1)y(t),$$
where μ is a positive parameter. $(0,0)$ is equilibrium. Show that this equilibrium is strictly stable by finding a strict Lyapunov function.
Solution: As we saw in lecture, a strict Lyapunov function for the linearised system is necessarily a strict Lyapunov function for the non-linear system, so we can replace the system above by its linearisation
$$x'(t) = y(t), \quad y'(t) = -x(t) - \mu y(t),$$
or, in matrix form,
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & -\mu \end{pmatrix}.$$
As shown in lecture,
$$V(x,y) = (\, x, y\,)\, B\, (\, x \quad y\,)$$
is a strict Lyapunov function if it is positive definite and
$$A^T B + BA + C = 0$$
for some positive definite C. It was also shown that for any such C there is a corresponding B. For simplicity let
$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
which is certainly positive definite. Name the entries of B,
$$B = \begin{pmatrix} p & q \\ q & r \end{pmatrix}.$$

Then
$$A^T B + BA + C = \begin{pmatrix} -2q+1 & p-\mu q - r \\ p - \mu q - r & 2q - 2\mu r + 1 \end{pmatrix}.$$
Setting the entries to zero and solving,
$$q = \frac{1}{2}, \quad r = \frac{1}{\mu}, \quad p = \frac{\mu}{2} + \frac{1}{\mu},$$
So
$$B = \begin{pmatrix} \frac{\mu}{2} + \frac{1}{\mu} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\mu} \end{pmatrix}.$$
This is indeed positive definite, so
$$V(x,y) = (\, x, y\,)\, B \,(\, x \quad y\,) = \left(\frac{\mu}{2} + \frac{1}{\mu}\right) x^2 + xy + \frac{1}{\mu} y^2$$
is a strict Lyapunov function.