MA 2326
Assignment 6
Due 19 March 2015

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1. Suppose that q and r are continuous functions on an interval I. Show that there exist solutions twice continuously differentiable function y_1 and y_2 of the differential equation

$$y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

such that

$$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

has no zeros in I.
Hint: This is a straightforward consequence of theorems proved in lecture. You just need to put them together in the right order.
Solution: Let

$$A(x) = \begin{pmatrix} 0 & 1 \\ -r(x) & -q(x) \end{pmatrix}.$$

As proved in lecture, there is a fundamental matrix W for A. Fix $x_0 \in I$ and define

$$y_1(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} W(x, x_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_2(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} W(x, x_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$y_1'(x) = \begin{pmatrix} 0 & 1 \end{pmatrix} W(x, x_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_2'(x) = \begin{pmatrix} 0 & 1 \end{pmatrix} W(x, x_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \det(W(x, x_)) = \exp\left(-\int_{x_0}^{x} q(t)\, dt\right).$$

The expression on the right hand side is never zero. For $j \in \{1, 2\}$ we have

$$\frac{d}{dx}\begin{pmatrix} y_j(x) \\ y_j'(x) \end{pmatrix} = A(x)\begin{pmatrix} y_j(x) \\ y_j'(x) \end{pmatrix}$$

and hence

$$y_j''(x) + q(x)y_j'(x) + r(x)y_j(x) = 0.$$

2. Conversely, suppose that y_1 and y_2 are twice continuously differentiable functions on an interval I and

$$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

has no zeros in I. Show that there are continuous functions p and q such that y_1 and y_2 are solutions of the differential equation

$$y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

on I.

Hint: This is *not* a straightforward consequence of theorems proved in lecture. You may find it useful to consider the matrix

$$\begin{pmatrix} y(x) & y_1(x) & y_2(x) \\ y'(x) & y_1'(x) & y_2'(x) \\ y''(x) & y_1''(x) & y_2''(x) \end{pmatrix}.$$

Solution: The general solution should be a linear combination of y_1 and y_2. In other words, there should be constants c_1 and c_2 such that

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Differentiating,

$$y'(x) = c_1y_1'(x) + c_2y_2'(x).$$

Differentiating again,

$$y''(x) = c_1y_1''(x) + c_2y_2''(x).$$

In matrix form, these equations are

$$\begin{pmatrix} y(x) & y_1(x) & y_2(x) \\ y'(x) & y_1'(x) & y_2'(x) \\ y''(x) & y_1''(x) & y_2''(x) \end{pmatrix}\begin{pmatrix} 1 \\ -c_1 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since we have a non-zero vector in the null space the determinant must be zero.

$$\det\begin{pmatrix} y(x) & y_1(x) & y_2(x) \\ y'(x) & y_1'(x) & y_2'(x) \\ y''(x) & y_1''(x) & y_2''(x) \end{pmatrix} = 0$$

or, expanding out the determinant,

$$\begin{aligned}&[y_1(x)y_2'(x) - y_1'(x)y_2(x)]\, y''(x)\\ &- [y_1(x)y_2''(x) - y_1''(x)y_2(x)]\, y'(x)\\ &+ [y_1'(x)y_2''(x) - y_1''(x)y_2'(x)]\, y(x) \; = 0.\end{aligned}$$

Dividing by $y_1(x)y_2'(x) - y_1'(x)y_2(x)$,

$$y''(x) + q(x)y'(x) + r(x)y(x) = 0,$$

where

$$q(x) = -\frac{y_1(x)y_2''(x) - y_1''(x)y_2(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}$$

and

$$r(x) = \frac{y_1'(x)y_2''(x) - y_1''(x)y_2'(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)}.$$

The quotients are continuous becauses the denominator, $w(x)$, was assumed to be non-zero. The argument just given shows that any y of the form

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

satisfies the differential equation just given. In particular y_1 and y_2 do, since they correspond to $c_1 = 1, c_2 = 0$ and $c_1 = 0, c_2 = 1$, respectively.

3. The second Painlevé equation is

$$y''(x) = 2y(x)^2 + xy(x) + \alpha.$$

α is a parameter.

(a) Show that for any α, x_0, y_0 and v_0 the equation has a unique maximally extended solution with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = v_0.$$

Hint: This is a straightforward consequence of theorems proved in the notes.
Solution: Setting

$$z(x) = \begin{pmatrix} y(x) & y'(x) \end{pmatrix}$$

the Painlevé equation is equivalent to the first order system

$$z'(x) = F(x, z(x))$$

where

$$F\left(x, \begin{pmatrix} y \\ v \end{pmatrix}\right) = \begin{pmatrix} v \\ 2y^3 + xy + \alpha \end{pmatrix}$$

F is defined and continuous throughout $\mathbf{R}^3$, as is the partial derivative matrix

$$\frac{\partial F}{\partial z} = \begin{pmatrix} 0 & 1 \\ 6y^2 + x & 0 \end{pmatrix},$$

so the usual existence and uniqueness theorem applies.

(b) Show if a maximally extended solution and its derivative are bounded then its interval of definition is all of $\mathbf{R}$.
Hint: The easiest way to do this is to assume a bound and an interval of definition and then use the quantitative version of the existence theorem, Theorem 6 of Chapter 1 of the notes, to derive a contradiction. Use the explicit form of the differential equation as little as possible.
Solution: If y and y' are bounded then there is a σ such that

$$\|z(x)\| < \sigma$$

for all x and choose an $s > \sigma$. Let x_0 be some point in the interval of definition of the solution and choose r large enough that $[x_0 - r, x_0 + r]$ is not contained in the interval of definition. By continuity there are M and N such that

$$\left\| \frac{\partial F}{\partial z}(x, z) \right\| \leq M$$

and

$$\|F(x, 0)\| \leq N$$

for all $x \in [x_0 - r, x_0 + r]$ and $z \in \overline{B}_s(0)$. Its not hard to find explicit M and N, but it's unnecessary. By the quantitative version of the existence theorem we can solve the equation starting from any initial point in the interval $x \in [x_0 - r, x_0 + r]$ and any initial values in $\overline{B}_s(0)$ for a distance of at least

$$\rho = \min\left(r, \frac{1}{M} \log\left(\frac{Ms + N}{M\sigma + N}\right)\right).$$

The explicit form is irrelevant. All we care about is that $\rho > 0$ and ρ doesn't depend on our initial conditions. Because $r, \rho > 0$

there is a positive integer $m > r/\rho$. We can then use the existence theorem m times to get a solution extending throughout $[x_0 - r, x_0 + r]$ starting from the values at x_0. But r was chosen in such a way that this extends beyond the interval of definition of the maximally extended solution. This contradicts the definition of maximal, so we have to reject the assumption that there is a bounded maximally extended solution which is not defined everywhere.