MA 2326
Assignment 5
Due 5 March 2015

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1. Show that

$$W(x,x_0) = \frac{1}{3}\begin{pmatrix} x^2x_0^{-2} + 2x^{-1}x_0 & x^2x_0^{-1} - x^{-1}x_0^2 \\ 2xx_0^{-2} - 2x^{-2}x_0 & 2xx_0^{-1} + x^{-2}x_0^2 \end{pmatrix}$$

is a fundamental matrix for

$$A(x) = \begin{pmatrix} 0 & 1 \\ 2x^{-2} & 0 \end{pmatrix}.$$

Use this to find the solution to the inhomogenous initial value problem

$$x^2y''(x) - 2y(x) = 1,$$

$$y(x_0) = y_0, \quad y'(x_0) = v_0$$

for $x > 0$.
Solution: Direct computation shows that

$$W'(x,x_0) = \frac{1}{3}\begin{pmatrix} 2xx_0^{-2} - 2x^{-2}y & 2xx_0^{-1} + x^{-2}x_0^2 \\ 2x_0^{-2} + 4x^{-3}y & 2x_0^{-1} - 2x^{-3}x_0^2 \end{pmatrix}$$

and

$$A(x)W(x,x_0) = \frac{1}{3}\begin{pmatrix} 2xx_0^{-2} - 2x^{-2}x_0 & 2xx_0^{-1} + x^{-2}x_0^2 \\ 2x_0^{-2} + 4x^{-3}x_0 & 2x_0^{-1} - 2x^{-3}x_0^2 \end{pmatrix}$$

so

$$W'(x,x_0) = A(x)W(x,x_0).$$

Also,

$$W(x,x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus W is a fundamental matrix for A. We therefore have

$$y(x) = (\,1 \quad 0\,)\, W(x,x_0) \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} + \int_{x_0}^{x} (\,1 \quad 0\,)\, W(x,t) \begin{pmatrix} 0 \\ 1/t^2 \end{pmatrix} dt,$$

or

$$\begin{aligned} y(x) &= \frac{(x^2x_0^{-2} + 2x^{-1}x_0)y_0 + (x^2x_0^{-1} - x^{-1}x_0^2)v_0}{3} \\ &\quad + \int_{x_0}^{x} \frac{x^2t^{-1} - x^{-1}t^2}{3} t^{-2}\, dt \\ &= \frac{(x^2x_0^{-2} + 2x^{-1}x_0)y_0 + (x^2x_0^{-1} - x^{-1}x_0^2)v_0}{3} \\ &\quad + \left[-\frac{x^2t^{-2} + 2x^{-1}t}{6} \right]_{t=x_0}^{t=x} \\ &= \frac{(x^2x_0^{-2} + 2x^{-1}x_0)y_0 + (x^2x_0^{-1} - x^{-1}x_0^2)v_0}{3} \\ &\quad + \frac{x^2x_0^{-2} + 2x^{-1}x_0 - 3}{6}. \end{aligned}$$

2. Find a non-zero quadratic polynomial which satisfies

$$(1 - x^2)y''(x) - 2xy'x + 6y(x) = 0$$

and then find a second, linearly independent, solution.
Solution: We want a solution of the form

$$y(x) = \alpha x^2 + \beta x + \gamma.$$

Subsituting into the differential equation,

$$2\alpha(1 - x^2) - 2x(2\alpha x + \beta) + 6(\alpha x^2 + \beta x + \gamma) = 0$$

The coefficient of x^2 on the left hand side is zero for any values of α, β and γ. To make the coefficients of x and 1 equal to zero as well we need

$$4\beta = 0, \quad 6\gamma - 2\alpha = 0.$$

The solutions we are looking for are therefore

$$y_1(x) = 3x^2 - 1$$

and its multiples. To find a second solution we solve the first order linear equations

$$(1 - x^2)w'(x) - 2xw(x) = 0$$

and

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) = w(x)$$

for w and y_2. The first has as its solution

$$w(x) = \frac{w(0)}{1-x^2}.$$

The second equation then becomes

$$(3x^2-1)y_2'(x) - 6xy_2(x) = \frac{w(0)}{1-x^2}.$$

This has the solution

$$y_2(x) = y_i(x) + y_h(x)$$

where

$$y_i(x) = y_2(0)\exp\left(\int_0^x \frac{6t}{3t^2-1}\,dt\right) = -y_2(0)(3x^2-1)$$

and

$$\begin{aligned}
y_i(x) &= \int_0^x \exp\left(\int_s^x \frac{6t}{3t^2-1}\,dt\right)\frac{w(0)}{(1-s^2)(3s^2-1)}\,ds \\
&= w(0)\int_0^x \frac{3x^2-1}{(3s^2-1)^2(1-s^2)}\,ds \\
&= \frac{w(0)(3x^2-1)}{4}\int_0^x \left(3\frac{3s^2+1}{(3s^2-1)^2} + \frac{1}{1-s^2}\right)ds \\
&= \frac{w(0)(3x^2-1)}{8}\int_0^x \left(\frac{1}{(s-1/\sqrt{3})^2} + \frac{1}{(s+1/\sqrt{3})^2} - \frac{1}{s-1} + \frac{1}{s+1}\right)ds \\
&= \frac{w(0)(3x^2-1)}{8}\left[\log\left(\frac{1+x}{1-x}\right) - \frac{6x}{3x^2-1}\right] \\
&= \frac{w(0)}{8}\left[\left(3x^2-1\right)\log\left(\frac{1+x}{1-x}\right) - 6x\right]
\end{aligned}$$

The simplest choice is $w(0) = 8$, $y(0) = 0$:

$$y_2(x) = (3x^2-1)\log\left(\frac{1+x}{1-x}\right) - 6x.$$