MA 2326
Assignment 4
Due 17 February 2015

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1. Compute $\exp(xA)$ for

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Note: The matrix A is real, so $\exp(xA)$ will be real as well. You should simplify your answer sufficiently that this is obvious. *Solution:* The characteristic polynomial is

$$p_A(\lambda) = \lambda^3 - 2\lambda^2 + 6\lambda - 4 = (\lambda - 1)(\lambda - 1 - i\sqrt{3})(\lambda - 1 + i\sqrt{3})$$

Luckily the roots are distinct, so we know that the Jordan canonical form is diagonal:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + i\sqrt{3} & 0 \\ 0 & 0 & 1 - i\sqrt{3} \end{pmatrix}.$$

Its exponential is

$$\begin{aligned} \exp(xC) &= \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^{(1+i\sqrt{3})x} & 0 \\ 0 & 0 & e^{(1-i\sqrt{3})x} \end{pmatrix} \\ &= e^x \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\sqrt{3}x) + i\sin(\sqrt{3}x) & 0 \\ 0 & 0 & \cos(\sqrt{3}x) - i\sin(\sqrt{3}x) \end{pmatrix}. \end{aligned}$$

To find V it suffices to find eigenvectors corresponding to the eigenvalues 1, $1+i\sqrt{3}$ and $1-i\sqrt{3}$. These are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix}.$$

so we can take

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \end{pmatrix}$$

Then

$$V^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{-1-i\sqrt{3}}{2} & \frac{-1+i\sqrt{3}}{2} \\ 1 & \frac{-1+i\sqrt{3}}{2} & \frac{-1-i\sqrt{3}}{2} \end{pmatrix},$$

$$\begin{aligned} \exp(xA) &= V \exp(xC) V^{-1} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} e^x \\ &\quad + \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} e^x \cos(\sqrt{3}x) \\ &\quad + \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} e^x \sin(\sqrt{3}x). \end{aligned}$$

A fair amount of matrix algebra has been skipped.

2. (a) An $n \times n$ matrix M if of rank k if and only if there is an $n \times k$ matrix P and a $k \times n$ matrix Q such that

$$M = PQ$$

and QP is invertible. Prove that in this case[1]

$$\exp(M) = I + P(QP)^{-1} \left(\exp(QP) - I \right) Q.$$

Note: The equation is always correct, but is only of practical use if k is much smaller than n. Do not attempt to use the equation $(QP)^{-1} = P^{-1}Q^{-1}$. This always fails if $k < n$, because neither P

[1]The two I's on the right hand side of this equation are not equal! The first occurence of I is the $n \times n$ identity matrix and the second occurence of I is the $k \times k$ identity matrix.

nor Q can be invertible in that case.
Solution:

$$\begin{aligned}
\exp(M) &= \sum_{m=0}^{\infty} \frac{1}{m!} M^m = I + \sum_{m=1}^{\infty} \frac{1}{m!} (PQ)^m \\
&= I + \sum_{m=1}^{\infty} \frac{1}{m!} P(QP)^{m-1} Q = I + P \left(\sum_{m=1}^{\infty} \frac{1}{m!} P(QP)^{m-1} \right) Q \\
&= I + P \left((QP)^{-1} \sum_{m=1}^{\infty} \frac{1}{m!} P(QP)^m \right) Q \\
&= I + P \left((QP)^{-1} (\exp(QP) - I) \right) Q \\
&= I + P(QP)^{-1} (\exp(QP) - I) Q.
\end{aligned}$$

(b) Use the preceding identity to compute $\exp(xA)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note: You may use the identity from the previous part even if you didn't succeed in proving it.
Solution: $M = xA = PQ$ where

$$P = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

Of course the x's can be absorbed into either factor.

$$QP = \begin{pmatrix} 4x \end{pmatrix},$$

so the identity from the previous part gives

$$\exp(xA) = I + P \begin{pmatrix} 4x \end{pmatrix}^{-1} \left(\exp\begin{pmatrix} 4x \end{pmatrix} - \begin{pmatrix} 1 \end{pmatrix}\right) Q = I + P \begin{pmatrix} \frac{e^{4x}-1}{4x} \end{pmatrix} Q$$

so

$$\exp(xA) = I + \begin{pmatrix} \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \end{pmatrix}$$

or

$$\exp(xA) = \begin{pmatrix} \frac{e^{4x}+3}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}+3}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}+3}{4} & \frac{e^{4x}-1}{4} \\ \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}-1}{4} & \frac{e^{4x}+3}{4} \end{pmatrix}.$$