MA 2326
Assignment 2
Due 3 February 2015

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1. Solve the initial value problem

$$y'(x) = \frac{2}{x} y(x) + 1$$

$$y(x_0) = y_0$$

where $x_0 \neq 0$.
Solution: Applying the general solution formula

$$y(x) = y_0 \exp\left(\int_{x_0}^{x} a(t)\,dt\right) + \int_{x_0}^{x} \exp\left(\int_{s}^{x} a(t)\,dt\right) b(s)\,ds$$

with

$$a(x) = \frac{2}{x}, \quad b(x) = 1$$

gives

$$y(x) = y_0 \frac{x^2}{x_0^2} + \int_{x_0}^{x} \frac{x^2}{s^2}\,ds = y_0 \frac{x^2}{x_0^2} + x^2 \left(\frac{1}{x_0} - \frac{1}{x}\right) = \left(\frac{y_0}{x_0^2} + \frac{1}{x_0}\right) x^2 - x.$$

2. Show that every linear inhomogeneous equation

$$y'(x) = a(x)y(x) + b(x)$$

possesses an integrating factor which is a function of x alone:

$$\mu(x, y) = h(x).$$

Solution: For μ to be an integrating factor we need

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0$$

where

$$p(x, y) = \mu(x, y)(a(x)y + b(x)), \quad q(x, y) = \mu(x, y).$$

Substituting, we need μ to satisfy the partial differential equation

$$(a(x)y + b(x))\frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial x} + a(x)\mu = 0.$$

If $\mu(x, y) = h(x)$ then this becomes

$$h'(x) + a(x)h(x) = 0.$$

This has a solution

$$h(x) = \exp\left(-\int_{x_0}^{x} a(t)\,dt\right).$$

Reversing the steps above, we see that this is indeed an integrating factor.

3. Given that

$$y(x) = x^5 - 3x^4 + 5x^3 - 7x^2 + 6x - 2$$

is a solution of the differential equation

$$y'(x) - y(x) + x^5 - 8x^4 + 17x^3 - 22x^2 + 20x - 8 = 0$$

find a solution with $y(0) = y_0$.
Solution: We have a solution of the inhomogenous equation. Since

$$y(x) = e^x$$

is a non-zero solution of the corresponding homogeneous equation

$$y'(x) - y(x) = 0$$

we can conclude that the general solution of

$$y'(x) - y(x) + x^5 - 8x^4 + 17x^3 - 22x^2 + 20x - 8 = 0$$

is

$$y(x) = ce^x + x^5 - 3x^4 + 5x^3 - 7x^2 + 6x - 2.$$

Substituting $x = 0$ and imposing the initial condtion leads to

$$y_0 = c - 2$$

so $c = y_0 + 2$ and

$$y(x) = (y_0 + 2)e^x + x^5 - 3x^4 + 5x^3 - 7x^2 + 6x - 2.$$

4. Use the integrating factor from Problem 2 to solve the initial value problem

$$y'(x) = a(x)y(x) + b(x),$$
$$y(x_0) = y_0$$

using the method given in lecture for integrable equations.
Note: We know already that the solution to the initial value problem is unique, so if your final answer isn't

$$y(x) = y_0 \exp\left(\int_{x_0}^{x} a(t)\,dt\right) + \int_{x_0}^{x} \exp\left(\int_{s}^{x} a(t)\,dt\right) b(s)\,ds$$

then you've done something wrong.
Solution: By the Poincaré Lemma there is a U such that

$$\frac{\partial U}{\partial x} = -p(x,y) = -\exp\left(-\int_{x_0}^{x} a(t)\,dt\right)(a(x)y + b(x))$$

and

$$\frac{\partial U}{\partial y} = q(x,y) = \exp\left(-\int_{x_0}^{x} a(t)\,dt\right)$$

The second equation implies

$$U(x,y) = \exp\left(-\int_{x_0}^{x} a(t)\,dt\right) y + f(x)$$

for some function f. The preceding equation then tells us that

$$f'(x) = -\exp\left(-\int_{x_0}^{x} a(t)\,dt\right) b(x)$$

and so

$$f(x) = -\int_{x_0}^{x} \exp\left(-\int_{x_0}^{s} a(t)\,dt\right) b(s)\,ds$$

plus a constant of integration, which we can take to be zero so that

$$U(x_0, y_0) = y_0.$$

Since U is invariant

$$U(x, y(x)) = U(x_0, y(x_0)) = U(x_0, y_0) = y_0.$$

Substituting the expression found earlier for U,

$$\exp\left(-\int_{x_0}^{x} a(t)\, dt\right) y(x) - \int_{x_0}^{x} \exp\left(-\int_{x_0}^{s} a(t)\, dt\right) b(s)\, ds = y_0.$$

Multiplying by $\exp\left(\int_{x_0}^{x} a(t)\, dt\right)$ and using the properties of the exponential function,

$$y(x) - \int_{x_0}^{x} \exp\left(\int_{x_0}^{x} a(t)\, dt - \int_{x_0}^{s} a(t)\, dt\right) b(s)\, ds = y_0 \exp\left(\int_{x_0}^{x} a(t)\, dt\right).$$

This can be simplified to

$$y(x) - \int_{x_0}^{x} \exp\left(\int_{s}^{x} a(t)\, dt\right) b(s)\, ds = y_0 \exp\left(\int_{x_0}^{x} a(t)\, dt\right).$$

Solving for $y(x)$ gives

$$y(x) = y_0 \exp\left(\int_{x_0}^{x} a(t)\, dt\right) + \int_{x_0}^{x} \exp\left(\int_{s}^{x} a(t)\, dt\right) b(s)\, ds,$$

as expected.