MA 2326
Assignment 1
Due 27 January 2015

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1. Solve initial value problem

$$y(x_0) = y_0, \quad y_0 \neq 0$$

for the separable equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

On what interval is the solution valid?
Note: Be careful with signs in this and the other problems.
Solution:

$$y\,dy = x\,dx.$$

Integrating,

$$\frac{1}{2}y^2 - \frac{1}{2}y_0^2 = \frac{1}{2}x^2 - \frac{1}{2}x_0^2$$

and

$$y^2 = y_0^2 - x_0^2 + x^2.$$

The solution is then

$$y(x) = \pm\sqrt{y_0^2 - x_0^2 + x^2}.$$

The sign is that of y_0, as we can see by considering $x = x_0$. The domain of validity depends on the sign of $y_0^2 - x_0^2$ and, possibly, of x_0. If $y_0^2 - x_0^2 > 0$ then the solution is valid for all x. Otherwise the solution is valid for $x > \sqrt{x_0^2 - y_0^2}$ if $x_0 > 0$ and for $x > \sqrt{x_0^2 - y_0^2}$ if $x_0 < 0$.

2. Solve initial value problem

$$y(x_0) = y_0, \quad x_0 \neq 0$$

for the separable equation

$$\frac{dy}{dx} = -\frac{y}{x}.$$

On what interval is the solution valid?
Solution:

$$\frac{dy}{y} = -\frac{dx}{x}.$$

Integrating,

$$\log y - \log y_0 = -\log x + \log x_0.$$

Exponentiating,

$$y(x) = \frac{y_0}{x_0}\frac{1}{x}.$$

Note that the argument above assumes $x, y > 0$, but the solution works for any non-zero choice of x_0. If $y_0 = 0$ then the solution is just $y(x) = 0$, which is correct. The domain of validity is $x > 0$ if $x_0 > 0$ and $x < 0$ if $x_0 < 0$.

3. Solve initial value problem

$$y(x_0) = y_0, \quad x_0 \neq 0$$

for the integrable equation

$$\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{3x^2y + y^3}.$$

On what interval is the solution valid?
Solution:

$$\frac{dy}{dx} = -\frac{p(x,y)}{q(x,y)}$$

where

$$p(x,y) = -x^3 - 3xy^2, \quad q(x,y) = 3x^2y + y^3.$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = -6xy + 6xy = 0,$$

so there is a function U such that

$$\frac{\partial U}{\partial x} = -p, \quad \frac{\partial U}{\partial y} = q.$$

This U will then be an invariant. From the first equation we see that

$$U(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + f(y)$$

for some function f. From the second equation we see that

$$f(y) = \frac{1}{4}y^4$$

plus some constant, which we can without loss of generality assume to be zero. So

$$U(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4.$$

We have, from the invariance of U and the initial conditions,

$$U(x, y(x)) = U(x_0, y(x_0)) = U(x_0, y_0).$$

Substituting the form found above for U,

$$\frac{1}{4}x^4 + \frac{3}{2}x^2y(x)^2 + \frac{1}{4}y(x)^4 = \frac{1}{4}x_0^4 + \frac{3}{2}x_0^2y_0^2 + \frac{1}{4}y_0^4$$

or

$$y(x)^4 + 6x^2y(x)^2 + x^4 - y_0^4 - 6x_0^2y_0^2 - x_0^4 = 0.$$

This should be thought of as a quadratic equation for $y(x)^2$. Its solutions are

$$y(x)^2 = -3x^2 \pm \sqrt{8x^4 + y_0^4 + 6x_0^2y_0^2 + x_0^4}.$$

Note that the quantity under the square root sign is always non-negative. To determine the sign we have to look at the initial conditions. Substuting $x = x_0$ gives

$$y(x_0)^2 = -3x_0^2 \pm \sqrt{(y_0^2 + 3x_0^2)^2} = -3x_0^2 \pm |y_0^2 + 3x_0^2|$$

The absolute value sign is redundant because $y_0^2+3x_0^2$ is never negative. This gives the correct value if we take the positive sign in front of the square root:

$$y(x)^2 = -3x^2 + \sqrt{8x^4 + y_0^4 + 6x_0^2y_0^2 + x_0^4}.$$

Then

$$y(x) = \pm\sqrt{-3x^2 + \sqrt{8x^4 + y_0^4 + 6x_0^2y_0^2 + x_0^4}}.$$

Here the sign has to agree with that of y_0, as we can again see by considering the case $x = x_0$.

For the solution we just found to make sense we need the quantity in the inner square root sign to be non-negative, which it always is, and the quantity inside the outer square root sign to be non-negative. For this we require

$$3x^2 \le \sqrt{8x^4 + y_0^4 + 6x_0^2y_0^2 + x_0^4}.$$

This is true if and only if

$$9x^4 \le 8x^4 + y_0^4 + 6x_0^2y_0^2 + x_0^4,$$

i.e. if

$$x^4 \le y_0^4 + 6x_0^2y_0^2 + x_0^4.$$

So we need

$$-\sqrt[4]{y_0^4 + 6x_0^2y_0^2 + x_0^4} \le x \le \sqrt[4]{y_0^4 + 6x_0^2y_0^2 + x_0^4}$$

Our solution however isn't differentiable at the endpoints, so these also need to be excluded. We have therefore a valid solution in the interval

$$-\sqrt[4]{y_0^4 + 6x_0^2y_0^2 + x_0^4} < x < \sqrt[4]{y_0^4 + 6x_0^2y_0^2 + x_0^4}.$$