MA 2326
Assignment 6
Due 19 March 2015

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1. Suppose that q and r are continuous functions on an interval I. Show that there exist solutions twice continuously differentiable function y_1 and y_2 of the differential equation

   $$y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

   such that

   $$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

   has no zeros in I.
   Hint: This is a straightforward consequence of theorems proved in lecture. You just need to put them together in the right order.

2. Conversely, suppose that y_1 and y_2 are twice continuously differentiable functions on an interval I and

   $$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

   has no zeros in I. Show that there are continuous functions p and q such that y_1 and y_2 are solutions of the differential equation

   $$y''(x) + q(x)y'(x) + r(x)y(x) = 0$$

   on I.
   Hint: This is *not* a straightforward consequence of theorems proved in lecture. You may find it useful to consider the matrix

   $$\begin{pmatrix} y(x) & y_1(x) & y_2(x) \\ y'(x) & y_1'(x) & y_2'(x) \\ y''(x) & y_1''(x) & y_2''(x) \end{pmatrix}.$$

3. The second Painlevé equation is

$$y''(x) = 2y(x)^2 + xy(x) + \alpha.$$

α is a parameter.

(a) Show that for any α, x_0, y_0 and v_0 the equation has a unique maximally extended solution with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = v_0.$$

Hint: This is a straightforward consequence of theorems proved in the notes.

(b) Show if a maximally extended solution and its derivative are bounded then its interval of definition is all of $\mathbf{R}$.
Hint: The easiest way to do this is to assume a bound and an interval of definition and then use the quantitative version of the existence theorem, Theorem 6 of Chapter 1 of the notes, to derive a contradiction. Use the explicit form of the differential equation as little as possible.