# Vortex Pairs via the Complex Hodograph Method

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We want to find the flow around a pair of vortices using the complex hodograph method, based on the fundamental equation

$$dz = \frac{dW}{\zeta},$$

where $z = x + iy$, $\zeta = u - iv$ and $W = U + iV$. It will be assumed that the two vortices are of the same strength and opposite direction, which is the only case in which the total energy is finite. $x$ and $y$ are the spatial coordinates and $u$ and $v$ are the corresponding components of velocity. $U$ and $V$ are the velocity potential and stream function.

We choose coordinates such that the $y$ axis goes through the centre of the vortices and we choose the origin to be equidistant from the two. Of the two such choices and we choose the one where the fluid in the upper half-plane is circulating in the anticlockwise direction and the fluid in the lower half-plane is circulating clockwise. There are two symmetries. Switching the signs of $x$ and $v$ or of $y$ and $v$ leaves the flow unchanged. For this reason we only need to find the flow in the first quadrant. $U$ and $V$ are determined only up to an additive constant, which we choose to be zero at the centre of symmetry.

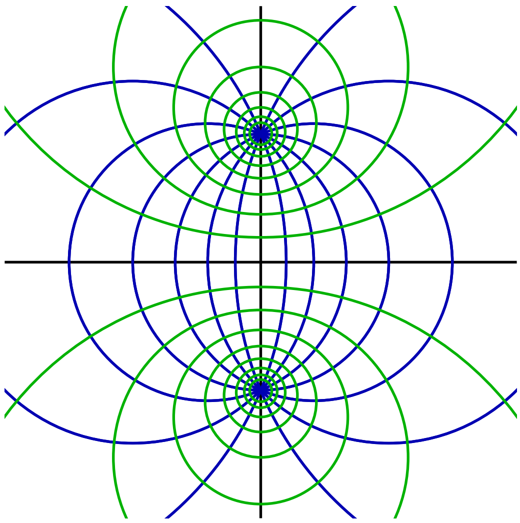
For reference, the corresponding Dirichlet flow is

$$\zeta = \frac{i\alpha}{z + iq} - \frac{i\alpha}{z - iq} = \frac{2\alpha q}{z^2 + q^2}$$

$$W = i\alpha \log\left(\frac{q - iz}{q + iz}\right).$$

$\alpha$ and $q$ are real constants describing the strength of the vortices and the distance between them.

The streamlines, lines of constant $V$ are shown in green on the following diagram. Lines of constant $U$ are shown in blue.



The branch cut in the logarithm, corresponding to rays beginning at the vortices and extending away from the centre of symmetry, isn't physically significant. Because there are closed streamlines there is no way to define a single-valued velocity potential everywhere. We could also avoid this issue by writing $z$ and $\zeta$ in terms of $W$:

$$z = q \tan\left(\frac{W}{2\alpha}\right), \quad \zeta = \frac{2\alpha}{q}\cos^2\left(\frac{W}{2\alpha}\right).$$

These are periodic functions with period $2\pi\alpha$.

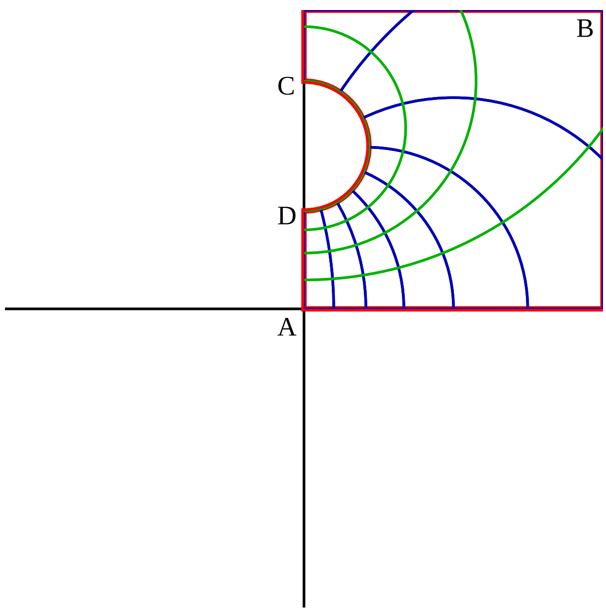
In view of Bernoulli's equation,

$$\frac{\rho}{2}|\zeta|^2 + p = p_0,$$

the Dirichlet flow above will have negative pressure near the vortices, regardless of the stagnation pressure $p_0$. In fact the pressure will tend to negative infinity at the vortices. To avoid this, we look for flows with cavities at the vortices. The cavities are bounded by streamlines, on which $V$ is constant, and the pressure is zero, so

$$|\zeta| = s = \sqrt{\frac{2p_0}{\rho}}.$$

Schematically, the picture is

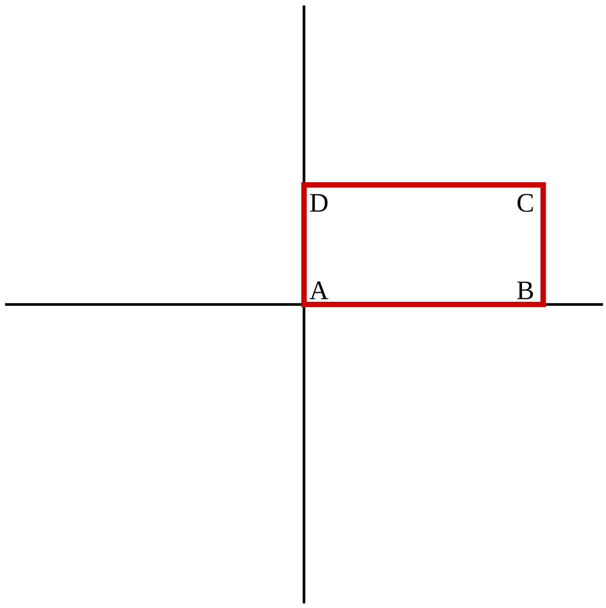



Only the first quadrant is shown. The part containing fluid is marked in red. The free streamline about the vortex is drawn as circular, although we will see that it is not, but all that matters here is that the $z$ picture should give an idea of the flow. It is the $\zeta$ and $W$ pictures which need to be accurate.

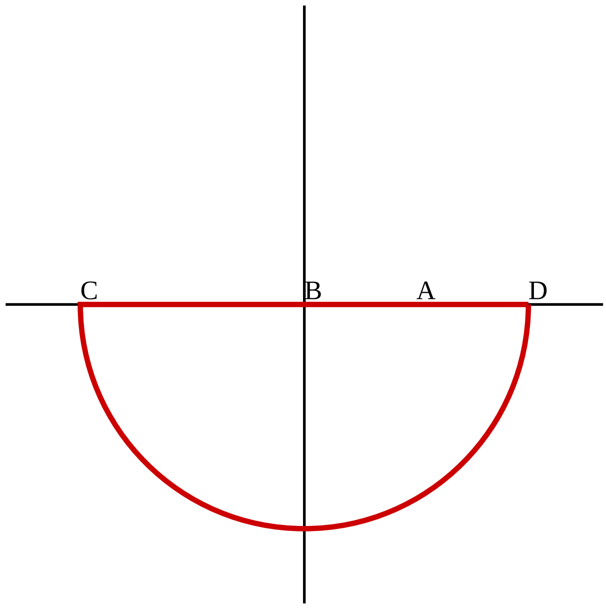
Four important points in the first quadrant are labelled. $A$ is the origin, which is the centre of symmetry of the whole picture. $B$ is meant to be located at infinity, but the diagram is merely schematic. $C$ and $D$ are points where the free streamline about the vortex intersects the $y$ axis. $C$ is the point farther from the centre of symmetry and $D$ is the point closer to the centre of symmetry. $A$ and $B$ are connected by streamlines, as are $C$ and $D$. $V$ is constant on these streamlines. $B$ and $C$ are connected by a line perpendicular to the direction of flow, as are $D$ and $A$, so $U$ is constant on these lines. These curves together bound the part of the first quadrant which contains fluid. The set of $W$ values corresponding to $z$ in the interior of the first quadrant therefore form a rectangle:

$$G = \{W \in \mathbf{C} \colon 0 < \operatorname{Re} W < K, 0 < \operatorname{Im} W < K'\}.$$

The origin is a corner of this rectangle, because we normalised $U$ and $V$ by setting them both equal to zero at infinity. The real constants $K$ and $K'$ are unknown. $K'$ at least has a physical meaning. $2\rho K'$ is the rate at which fluid flows between the vortices. The rectangle is shown below, with labels $A$, $B$, $C$ and $D$ corresponding to those in the previous diagram.




On the streamline connecting $A$ and $B$ the vertical component of velocity is always zero and the horizontal component is always positive, decreasing from some $\sigma < s$, which is unknown, to zero. On the ray connecting $B$ to $C$ the vertical component of velocity is again zero and the horizontal component is negative, decreasing from 0 to $-s$. On the free streamline connecting $C$ to $D$ the vertical component of velocity is always positive and, since this streamline borders a cavity, the speed is always $s$. The horizontal component of velocity increases from $-s$ to $s$. On the line segment connecting $D$ to $A$ the vertical component of velocity is always zero and the horizontal component is positive, decreasing from $s$ to $\sigma$. Tracing the corresponding values of $\zeta$ we obtain the following diagram, with labels corresponding to those in the previous two diagrams.




Points $z$ in the interior of the first quadrant must correspond to values of $\zeta$ in the interior of the the lower half-disc of radius $s$,

$$H = \{z \in \mathbf{C}\colon |\zeta| < s, \operatorname{Im}\zeta < 0\}.$$

By the Riemann Mapping Theorem there is an invertible analytic map from $R$ to $H$. We need this map to take the points labelled $A$, $B$, $C$ and $D$ in the $W$ diagram to those with the same labels in the $\zeta$ diagram. This is not, in general, possible. We will need a certain relation to hold between the dimensionless ratios $\sigma/s$ and $K'/K$.

The Riemann Mapping Theorem, unfortunately, does not give us an explicit mapping. The most useful tool for constructing one is the Schwarz Reflection Principle:

> If $\Omega \in \mathbf{C}$ is symmetric under reflection about the real axis, $f$ is analytic in $\Omega$ and
>
> $$f(\overline{z}) = \overline{f(z)}$$

then $f$ is real on the real axis. Conversely, if $f$ is analytic in the part of $\Omega$ in the upper half plane, including the real line, and is real on the real line then $f$ can be extended to all of $\Omega$ as an analytic function via

$$f(\overline{z}) = \overline{f(z)}.$$

There is nothing special about the real line, either in the domain or image. We can replace it with any other line, replacing the mapping $z \to \overline{z}$ with reflection about that line. We need not use the same line for the domain and image, or, equivalently, the same reflection on both sides of the equation. We can also replace either or both lines by circles, replacing the reflection $z \to \overline{z}$ by inversion with respect to that circle. The generalised Schwarz Reflection Principle is easier to apply than to state precisely. To handle inversions correctly we need to use the "Riemann sphere" $\Sigma = \mathbf{C} \cup \{\infty\}$ in place of $\mathbf{C}$.

Suppose that $C$ is a circle or line and $R: \Sigma \to \Sigma$ is inversion or reflection with respect to that circle or line. Suppose that $\Omega \subset \Sigma$ satisfies $R(\Omega) = \Omega$. Suppose that $C'$ is a (possibly) different circle or line and $R'$ the corresponding reflection or inversion. Suppose $f: \Omega \to \Sigma$ is meromorphic. If

$$f(R(z)) = R'(f(z))$$

then $f(C) \subset C'$. Conversely, if $f$ is defined and meromorphic on the part of $\Omega$ to one side of and including $C$, and satisfies $f(C) \subset C'$, then extending $f$ to all of $\Omega$ via

$$f(R(z)) = R'(f(z))$$

gives an analytic function.

In our case we can apply the generalised Schwarz Reflection Principle to each of the four sides of the rectangle $Q$. In the $W$ plane we have the four reflections

$$R_{AB}(W) = \overline{W}, \quad R_{BC}(W) = 2K - \overline{z}$$

$$R_{CD}(W) = 2iK' + \overline{z} \quad R_{DA}(W) = -\overline{W}.$$

These correspond the reflections/inversions

$$R'_{AB}(\zeta) = \overline{\zeta}, \quad R'_{BC}(\zeta) = \overline{\zeta},$$

$$R'_{CD}(\zeta) = s^2/\overline{\zeta}, \quad R'_{DA}(\zeta) = \overline{\zeta}$$

in the $\zeta$ plane. In each case the reflection/inversion is found by looking at the corresponding segment/arc in appropriate diagram. By the generalised Schwarz Reflection Principle the mapping from $\zeta: G \to H$ has an extension satisfying

$$\zeta(R_{PQ}(W)) = R'_{PQ}(\zeta(W))$$

for any $PQ$ in the set $AB$, $BC$, $CD$, $DA$. We can repeat this construction, with a different choice of in place of $PQ$:

$$\begin{aligned}\zeta(R_{ST}(R_{PQ}(W))) &= R'_{ST}(\zeta(R_{PQ}(W))) \\ &= R'_{ST}(R'_{PQ}(\zeta(W))).\end{aligned}$$

There are several interesting choices of $PQST$. Choosing $PQST = DABC$ gives

$$\zeta(W + 2K) = \zeta(W),$$

so $\zeta$ is periodic. Choosing $PQST = ABCD$ gives

$$\zeta(W + 2iK') = s^2/\zeta(W).$$

Applying this twice,

$$\zeta(W + 4iK') = \zeta(W),$$

so $\zeta$ is doubly periodic. In addition to the real period $2K$ it has a purely imaginary period $4iK'$. Doubly periodic meromorphic functions are called elliptic, and there is an extensive theory of such functions. Finally, choosing $PQST = ABBC$,

$$\zeta(-W) = \zeta(W),$$

so $\zeta$ is an even function. It has a zero at the point $K$, and must, because of the equation $\zeta(W+2iK') = s^2/\zeta(W)$, have a pole at $K+2iK'$. Because of the periodicity, adding any integer multiple of $2K$ or $4iK'$ to this zero or pole gives another zero or pole. There are no others. It follows from the periodicity and evenness of $\zeta$ that these zeroes and poles are of even order. A more careful analysis shows that the order is two. There is, up to multiplication by a constant, only one function with these properties. We can use

$$\zeta(0) = \sigma$$

or

$$\zeta(iK') = s$$

to fix that multiplicative constant, but not both. That is the origin of the relation between $\sigma/s$ and $K'/K$ discussed earlier. In terms of the Weierstrass $\wp$ function,

$$\wp(z, \omega_1, \omega_2) = z^{-2} \sum_{(m,n)\in\mathbf{Z}^2-\{0,0\}} \left((z - m\omega_1 - n\omega_2)^{-2} - (m\omega_1 + n\omega_2)^{-2}\right)$$

the function $\zeta$ can be written explicitly as

$$\zeta(W) = \sigma \frac{\wp(W + K + 2iK', 2K, 4iK') - \wp(2iK', 2K, 4iK')}{\wp(K + 2iK', 2K, 4iK') - \wp(2iK', 2K, 4iK')}.$$

One can then find $z(W)$ by integrating

$$dz = \frac{dW}{\zeta(W)}.$$

The result is not an elliptic function, but it is expressible in terms of known functions. The streamlines are then found by fixing $V$ and letting $U$ vary. The relation between $\sigma/s$ and $K'/K$ is

$$\frac{\sigma}{s} = \frac{\wp(K + 2iK', 2K, 4iK') - \wp(2iK', 2K, 4iK')}{\wp(K + 3iK', 2K, 4iK') - \wp(2iK', 2K, 4iK')}.$$

It is fairly straightforward to show, from the definition of $\wp$, that the right hand side depends on $K$ and $K'$ only through the ratio $K'/K$.