1. (a) (6 points) Write down the characteric equations for a general fully nonlinear scalar first order partial differential equation in two independent variables. Apply the general formula in the special case of Burgers' equation

$$u_t + uu_x = 0.$$

   (b) (6 points) Find the characteristic curves for Burgers' equation and use these to derive the "solution formula" for the initial value problem

$$u(0, x) = f(x).$$

   (c) (8 points) Solve in the particular case $f(x) = x^2$.

2. (a) (6 points) The Euler-Lagrange equations corresponding to the Lagrangian

$$L(x, y, z, z_x, z_y) = \frac{z_{xx} + z_{yy}}{z} - \frac{z_x^2 + z_y^2}{z^2}.$$

   Interpret your result.

   (b) (5 points) Find the Euler-Lagrange equations corresponding to the Lagrangian

$$L(x, y, z, z_x, z_y) = \sqrt{1 - z_x^2 - z_y^2}.$$

   (c) (9 points) Find the infinitesimal generators for the one parameter families of symmetries of the Lagrangian L in the previous part.

      i. $(x, y, z) \to (x + s, y, z)$,

      ii. $(x, y, z) \to (x, y, z + s)$, and

      iii. $(x, y, z) \to (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z)$.

   Also, find the corresponding conserved currents.

3. (a) (2 points) To which class of differential equations is the finite element method applicable?

   (b) (2 points) What is numerical instability?

   (c) (8 points) Set up a finite difference scheme for the harmonic oscilator equation

$$u_{xx} + \omega^2 u = 0.$$

   on grid with spacing h. For which h is your scheme numerically unstable?

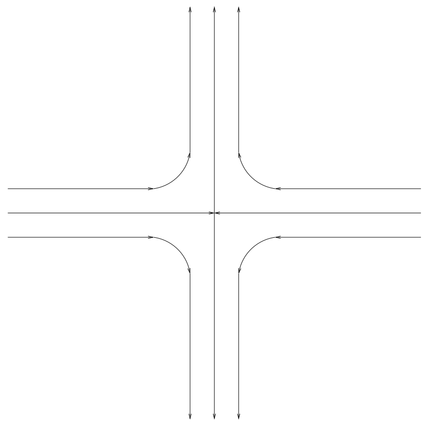
(d) (8 points) Set up a finite element scheme for the differential equation

$$y_{xx} = y^2 + y$$

in an interval $[a, b]$ with continuous piecewise linear elements.

4. (a) (8 points) What is the *complex hodograph method*? To which types of fluid flows does it apply?

   (b) (12 points) A pair of free streams, of equal flow rate and oppositely directed, meet at a stagnation point and split again into a pair of outgoing streams perpendicular to the incoming streams, as in the figure below.



   Assume the flow satisfies all the conditions you identified in the previous part and use the complex hodograph method to find the free streamlines. You may make any convenient choice of units, axes, etc. to simplify the problem as long as you state your choices clearly.